Tail Asymptotics for Cumulative Processes Sampled at Heavy-Tailed Random Times with Applications to Queueing Models in Markovian Environments*

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Abstract: This paper studies the tail asymptotics of a cumulative process $\{B(t); t \ge 0\}$ sampled at heavy-tailed random times T, where T has a dominant impact on the asymptotic behavior of $\mathsf{P}(B(T) > x)$. We establish several sufficient conditions for the asymptotic equality $\mathsf{P}(B(T) > bx) \sim \mathsf{P}(M(T) > bx) \sim \mathsf{P}(T > x)$ as $x \to \infty$, where $M(t) = \sup_{0 \le u \le t} B(u)$ and b is a certain positive constant. We also apply the obtained results to the subexponential asymptotics of the loss probability of a single-server finite-buffer queue with an on/off arrival process in a Markovian environment.

Keywords: Queue, Markovian environment, batch Markovian arrival process (BMAP), tail asymptotics, sampling, cumulative process, heavy-tailed

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1 Introduction

The main purpose of this paper is to provide mathematical tools for investigating the heavy-tailed asymptotic behavior of queueing models in Markovian environments. Many researchers have studied the heavy-tailed asymptotics of the random sum of random variables (r.v.s), and several interesting results have been reported in the literature. However, those results can not be applied directly to queueing models in Markovian environments, such as queues with

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batch Markovian arrival processes (BMAPs) [31] and general semi-Markovian arrival processes. Therefore, in this paper, we construct a framework for studying the heavy-tailed asymptotics for such queueing models.

Let $\{B(t); t \ge 0\}$ denote a (possibly delayed) cumulative process on $\mathbb{R} := (-\infty, \infty)$, where $|B(0)| < \infty$ with probability one (w.p.1) (see, e.g., [41]). Let τ_n $(n = 0, 1, \dots)$ denote the nth regenerative point of $\{B(t)\}$. Let $\Delta B_n = B(\tau_n) - B(\tau_{n-1})$ and $\Delta\tau_n = \tau_n - \tau_{n-1}$ for $n = 0, 1, \dots$, where $\tau_{-1} = 0$. The ΔB_n's (resp. $\Delta\tau_n$'s) $(n = 1, 2, \dots)$ are independent and identically distributed (i.i.d.) and independent of ΔB_0 (resp. $\Delta\tau_0$). Let $\Delta B_n^* = \sup_{\tau_{n-1} \le t \le \tau_n} B(t) - B(\tau_{n-1})$ for $n = 0, 1, \dots$. Clearly, $\Delta B_n^* \ge \Delta B_n$ for $n = 0, 1, \dots$, and the ΔB_n^*'s $(n = 1, 2, \dots)$ are i.i.d. and independent of ΔB_0^*. Further, let $M(t) = \sup_{0 \le u \le t} B(u)$ for $t \ge 0$.

Throughout this paper, we assume that

$$\begin{aligned} &\mathsf{P}(0 \le \Delta\tau_n < \infty) = \mathsf{P}(0 \le \Delta B_n^* < \infty) = 1 \;\; (n = 0, 1), \\ &\mathsf{E}[|\Delta B_1|] < \infty, \quad 0 < \mathsf{E}[\Delta\tau_1] < \infty, \quad b := \frac{\mathsf{E}[\Delta B_1]}{\mathsf{E}[\Delta\tau_1]} > 0. \end{aligned} \tag{1.1}$$

If $\mathsf{E}[\sup_{\tau_0 \le t \le \tau_1} |B(t) - B(\tau_0)|] < \infty$, then the strong law of large numbers (SLLN) for $\{B(t)\}$ holds, i.e., $\lim_{t\to\infty} B(t)/t = b$ w.p.1 (see, e.g., [3, Chapter VI, Theorem 3.1]).

This paper studies the tail asymptotics of $\mathsf{P}(B(T) > x)$, where T is a nonnegative r.v. for *sampling times* of $\{B(t)\}$. This study is motivated by the tail asymptotics for queueing models in Markovian environments. We now present a typical example of the application of the asymptotic results for $\mathsf{P}(B(T) > x)$. Consider a stationary BMAP/GI/1 queue under the first-in-first-out (FIFO) discipline. Suppose T is the service time of one customer, and $B(t)$ is the total number of BMAP arrivals in the interval $(0, t]$, which is a cumulative process. It is known that the subexponential asymptotics of the queue length distribution is closely related to the tail asymptotics of the number of arrivals in the service time of one customer (see [32, 42]), which is just an application of the asymptotic result for $\mathsf{P}(B(T) > x)$ with T being independent of $\{B(t)\}$.

The main contribution of this paper is to establish sufficient conditions for

$$\mathsf{P}(B(T) > bx) \overset{x}{\sim} \mathsf{P}(M(T) > bx) \overset{x}{\sim} \mathsf{P}(T > x), \tag{1.2}$$

where for any functions f and g, $f(x) \overset{x}{\sim} g(x)$ represents $\lim_{x\to\infty} f(x)/g(x) = 1$ (if the limit holds). There have been several related results. Before stating them, we need some definitions:

Definition 1.1 A nonnegative r.v. X and its distribution function (d.f.) F_X belong to the long-tailed class $\mathcal{L}$ if $\mathsf{P}(X > x) > 0$ for all $x \ge 0$ and $\mathsf{P}(X > x + y) \overset{x}{\sim} \mathsf{P}(X > x)$ for some (thus all) $y > 0$.

Definition 1.2 (i) pth-order long-tailed class $\mathcal{L}^p$ ($p \ge 1$): A nonnegative r.v. X and its d.f. F_X belong to class $\mathcal{L}^p$ if $X^{1/p} \in \mathcal{L}$. Further, if $X \in \mathcal{L}^{1/\theta}$ (resp. $F_X \in \mathcal{L}^{1/\theta}$) for any $0 < \theta \le 1$, we write $X \in \mathcal{L}^\infty$ (resp. $F_X \in \mathcal{L}^\infty$) and call X (resp. F_X) infinite-order long-tailed.

(ii) Consistent variation class $\mathcal{C}$: A nonnegative r.v. X and its d.f. F_X belong to class $\mathcal{C}$ if $\overline{F}_X(x) > 0$ for all $x \ge 0$ and

$$\lim_{v \downarrow 1} \liminf_{x \to \infty} \frac{\overline{F}_X(vx)}{\overline{F}_X(x)} = 1 \ \text{ or equivalently, } \ \lim_{v \uparrow 1} \limsup_{x \to \infty} \frac{\overline{F}_X(vx)}{\overline{F}_X(x)} = 1,$$

where $\overline{F}_X(x) = 1 - F_X(x)$ for all $x \in \mathbb{R}$.

Remark 1.1 Clearly, $\mathcal{L}^1 = \mathcal{L}$ and $\mathcal{L}^2$ is equivalent to the class of square-root insensitive distributions (see Lemma 1 in [25]). Typical examples of the distributions in $\mathcal{L}^p$ are the following:

(i) $\mathsf{P}(X > x) \overset{x}{\sim} \exp\{-x^\alpha\}$, where $0 < \alpha < 1/p$.

(ii) $\mathsf{P}(X > x) \overset{x}{\sim} \exp\{-x^{1/p}/(\log x)^\gamma\}$, where $\gamma > 0$.

Remark 1.2 The class $\mathcal{C}$ is sometimes called *intermediate regular variation class*. As for the inclusion relation between class $\mathcal{C}$ and higher-order long-tailed classes, it holds that $\mathcal{C} \subset \mathcal{L}^\infty \subset \mathcal{L}^p \subset \mathcal{L}$ for all $p > 1$ (see also Remark A.1 and Lemma A.5).

We now give a review of the related works, which are classified into two cases: (i) T is independent of $\{B(t)\}$; and (ii) T is not necessarily independent of $\{B(t)\}$. The former is called *independent-sampling case*, and the latter is called *non-independent-sampling case*. The non-independent-sampling case includes a case where T is a stopping time with respect to $\{B(t)\}$.

To the best of our knowledge, there are a few results in the non-independent-sampling case. Robert and Segers [37] consider a special case where

$$B(t) = \sum_{n=1}^{\lfloor t \rfloor} X_n \ \text{ with the } X_n\text{'s being i.i.d. nonnegative r.v.s.} \tag{1.3}$$

For the special case, they present the following results:

Proposition 1.1 (Theorem 4.1 in Robert and Segers [37]) *Suppose $X, X_1, X_2, \dots$ are i.i.d. nonnegative r.v.s. Further, suppose (i) T satisfies*

$$\lim_{x \to \infty} \frac{\mathsf{P}(T > x + ya(x))}{\mathsf{P}(T > x)} = e^{-y}, \qquad y \in \mathbb{R}, \tag{1.4}$$

for some function $a(x)$ ($x \ge 0$) such that $\lim_{x\to\infty} x^{2/3}/a(x) = 0$; and (ii) $\mathsf{E}[e^{\gamma X}] < \infty$ for some $\gamma > 0$. Then

$$\mathsf{P}(X_1 + \cdots + X_{\lfloor T \rfloor} > \mathsf{E}[X]x) \overset{x}{\sim} \mathsf{P}(T > x). \tag{1.5}$$

Proposition 1.2 (Theorem 3.1 in Robert and Segers [37]) *Suppose $X, X_1, X_2, \dots$ are i.i.d. nonnegative r.v.s. Then (1.5) holds if (i) $T \in \mathcal{C}$; (ii) $\mathsf{E}[X^\gamma] < \infty$ for some $\gamma > 1$; and (iii) $x\mathsf{P}(X > x) = o(\mathsf{P}(T > x))$, where for any functions f and g, $f(x) = o(g(x))$ represents $\lim_{x\to\infty} f(x)/g(x) = 0$ (if the limit holds).*

Next, we review the independent-sampling case. Asmussen et al. [4] prove

$$\mathsf{P}(B(T) > bx) \overset{x}{\sim} \mathsf{P}(T > x), \tag{1.6}$$

where $\{B(t)\}$ is assumed to be a renewal counting process. They also show that an important necessary condition for (1.6) is that $\sqrt{T}$ is heavy-tailed, i.e., $\mathsf{P}(T > x) = e^{-o(\sqrt{x})}$. Foss and Korshunov [15] extend Asmussen et al. [4]'s results to more general renewal counting processes and sampling times (see Theorem 5.1 therein). As for the general cumulative process, Jelenković et al. [25] present the following:

Proposition 1.3 (Proposition 3 in Jelenković et al. [25]) *Suppose T is independent of $\{B(t); t \ge 0\}$. Further, suppose (i) $T \in \mathcal{L}^2$ (i.e., $\sqrt{T} \in \mathcal{L}$); (ii) $\mathsf{E}[(\Delta\tau_1)^2] < \infty$ and $\Delta B_n \ge 0$ ($n = 0, 1$) w.p.1; and (iii) $\mathsf{E}[\exp\{\eta\sqrt{\Delta B_n^*}\}] < \infty$ ($n = 0, 1$) for some $\eta > 0$. Then (1.2) holds.*

There are some results on the random sum of r.v.s in the independent-sampling case.

Proposition 1.4 (Theorem 1.2 in Aleškevičienė et al. [1]) *Suppose $X, X_1, X_2, \dots$ are i.i.d. nonnegative r.v.s and T is independent of $\{X_n; n = 1, 2, \dots\}$. Further, suppose (i) $T \in \mathcal{C}$; (ii) $\mathsf{E}[X] < \infty$; and (iii) $\mathsf{E}[T] < \infty$ and $\mathsf{P}(X > x) = o(\mathsf{P}(T > x))$. Then (1.5) holds.*

Lin and Shen [30] extend Proposition 1.4 to the case where the X_n's are asymptotically quadrant sub-independent and identically distributed (see Theorem 2.1 (I) therein). Robert and Segers [37] present a theorem similar to Proposition 1.4 (see Theorem 3.2 therein). The theorem states that (1.5) requires $\mathsf{E}[X^r] < \infty$ for some $r > 1$, which is more restrictive than condition (ii) of Proposition 1.4. However, the theorem also presents a sufficient condition for (1.5) with $\mathsf{E}[T] = \infty$, which is described in the following:

Proposition 1.5 (Theorem 3.2 in Robert and Segers [37]) *Suppose $X, X_1, X_2, \dots$ are i.i.d. nonnegative r.v.s and T is independent of $\{X_n; n = 1, 2, \dots\}$. Further, suppose (i) $T \in \mathcal{C}$ and $\mathsf{E}[T] = \infty$; (ii) $\mathsf{E}[X^r] < \infty$ for some $r > 1$; and (iii) $\mathsf{E}[T \cdot \mathbb{1}(T \le x)] = O(x^q \mathsf{P}(T > x))$ for some $1 \le q < r$, where $\mathbb{1}(\chi)$ denotes the indicator function of an event χ. Then (1.5) holds.*

In what follows, we summarize the contributions of this paper. For the non-independent-sampling case, we assume that $\{B(t)\}$ is nondecreasing with t (e.g., $\{B(t)\}$ is a counting process of BMAP arrivals). Under this assumption, we present two theorems: Theorems 3.1 and 3.2, which are extensions of Propositions 1.1 and 1.2, respectively, to the general cumulative process, and which are still more general than the propositions even if (1.3) holds, i.e., $B(T)$ is reduced to the random sum of i.i.d. nonnegative r.v.s.

As for the independent-sampling case, we do not necessarily assume that $\{B(t)\}$ is nondecreasing with t, which means that ΔB_n can take negative values. We first present two theorems: Theorems 3.3 and 3.4. Theorem 3.3 provides a weaker sufficient condition for (1.2) than that in Proposition 1.3. Theorem 3.4 is an extension of Propositions 1.4 and 1.5 to the general cumulative process. However, unfortunately, when $B(t)$ satisfies (1.3), one of the conditions of Theorem 3.4 is more restrictive than the corresponding ones of Propositions 1.4 and 1.5. Thus, instead of the general cumulative process, we next consider a special case where $B(t) = B(\lfloor t \rfloor)$ for all $t \ge 0$ and $\{B(n); n = 0, 1, \dots\}$ is the additive component of a discrete-time Markov additive process (see, e.g., [3, Chapter XI, Section 2]), which implies that $B(T)$ is the random sum of r.v.s with Markovian correlation. Under this assumption, we prove another two theorems that completely include Propositions 1.4 and 1.5 as special cases. Further, the theorems are readily extended to a continuous-time Markov additive process.

As mentioned before, the results in the independent-sampling case can be applied to the stationary BMAP/GI/1 queue. Indeed, Masuyama et al. [32] present some formulas for the queue length and waiting time distributions by using Proposition 1.3, and thus it is expected that the results presented in this paper yield new formulas. In addition, unlike the existing results, our theorems for the independent-sampling case do not require the monotonicity of $\{B(t)\}$ and thus they can be also applied to an M/GI/1 queue with negative customers (see, e.g., [5]) and its extension to BMAP arrivals. For lack of space, we omit the details.

To demonstrate the utility of the main results in this paper, we discuss their application to the subexponential asymptotics of the loss probability of a discrete-time single-server queue with a finite buffer fed by an on/off arrival process in a Markovian environment. In the on/off arrival process, the lengths of on-periods (resp. off-periods) are i.i.d. with a general distribution, and arrivals in each on-period follow a discrete-time BMAP started with some initial distribution at the beginning of the on-period. We call the arrival process *on/off batch Markovian arrival process (ON/OFF-BMAP)*, which is a generalization of the batch-on/off process [17] and is closely related to a platoon arrival process (PAP) [2, 8] (see also Remarks 4.1 and 4.2). For analytical convenience, we assume that service times are all equal to the unit of time. The queueing model is denoted by (ON/OFF-BMAP)/D/1/K in Kendall's notation. For this queue, we derive subexponential asymptotic formulas for the loss probability, combining our results with the existing ones on a finite GI/GI/1 queue [22].

The rest of this paper is organized as follows. Section 2 summarizes preliminary results. Section 3 presents the main results, and Section 4 discusses their application to the (ON/OFF-BMAP)/D/1/K queue. Appendix A is devoted to preliminary lemmas. The proofs of all the lemmas and theorems are given in Appendices B and C.

2 Preliminaries

For later use, we introduce the following notations. Let C (resp. c) denote a special symbol representing a sufficiently large (resp. small) positive constant, which takes an appropriate value according to the context. Thus C (resp. c) can take different values in different places. For example, C in a place may be equal to $C+1$, $2C$ and C^2, etc. in other places. For any r.v. U in $\mathbb{R}$, let $U^+ = \max(U, 0)$ and let F_U denote the d.f. of U, i.e., $F_U(x) = \mathsf{P}(U \le x)$ for $x \in \mathbb{R}$, which is assumed to be right-continuous. Further, let $\overline{F}_U = 1 - F_U$ and $Q_U = -\log \overline{F}_U$. The latter is called the cumulative hazard function of U. Finally, for any nonnegative functions f and g, $f(x) = O(g(x))$, $f(x) \lesssim_x g(x)$ and $f(x) \gtrsim_x g(x)$ represent

$$\limsup_{x\to\infty} f(x)/g(x) < \infty, \qquad \limsup_{x\to\infty} f(x)/g(x) \le 1, \qquad \liminf_{x\to\infty} f(x)/g(x) \ge 1,$$

respectively.

2.1 Subexponential distributions

We begin with the definition of the subexponential class.

Definition 2.1 A nonnegative r.v. X and its d.f. F_X belong to the subexponential class $\mathcal{S}$ if $\mathsf{P}(X > x) > 0$ for all $x \ge 0$ and $\mathsf{P}(X_1 + X_2 > x) \overset{x}{\sim} 2\mathsf{P}(X > x)$, where X_1 and X_2 are independent copies of X.

Remark 2.1 The class $\mathcal{S}$ was first introduced by Chistyakov [10], and it was shown that $\mathcal{S}$ is a strictly subclass of the class $\mathcal{L}$, i.e., $\mathcal{S} \subset \mathcal{L}$ (see [36]).

Next we introduce the subexponential concave class $\mathcal{SC}$, which is a subclass of the subexponential class $\mathcal{S}$. The $\mathcal{SC}$ class plays a key role in establishing large deviation bounds for a cumulative process.

Definition 2.2 A nonnegative r.v. X, and its d.f. F_X and cumulative hazard function Q_X belong to the subexponential concave class $\mathcal{SC}$ if the following are satisfied: (i) Q_X is eventually concave; (ii) $\lim_{x\to\infty} Q_X(x)/\log x = \infty$; and (iii) there exist some $0 < \alpha < 1$ and $x_0 > 0$ such that $Q_X(x)/x^\alpha$ is nonincreasing for all $x \ge x_0$, i.e.,

$$\frac{Q_X(x)}{Q_X(u)} \le \left(\frac{x}{u}\right)^\alpha, \qquad x \ge u \ge x_0. \tag{2.1}$$

We may use the notation $\mathcal{SC}_\alpha$ to emphasize the parameter α.

Remark 2.2 Typical examples of the cumulative hazard function in $\mathcal{SC}$ are (i) $(\log x)^\gamma x^\alpha$ and (ii) $(\log x)^\beta$ for sufficiently large x, where $0 < \alpha < 1$, $\beta > 1$ and $\gamma \in \mathbb{R}$. See Appendix A.2 for further remarks.

The following lemma is used to prove Theorems 3.1 and 3.3.

Lemma 2.1 *Assume* $\mathsf{E}[(\Delta B_1)^2] < \infty$.

(i) If $\mathsf{E}[(\Delta\tau_1)^2] < \infty$ *and* $\mathsf{E}[\exp\{Q(\Delta B_n^*)\}] < \infty$ *(*$n = 0, 1$*) for some* $Q \in \mathcal{SC}$*, then for all* $x, u \geq 0$*,*

$$\mathsf{P}\left(\sup_{0\le t\le x}\{B(t) - bt\} > u\right) \le C\left(e^{-cu^2/x} + e^{-cx} + xe^{-cQ(u)}\right).$$

(ii) Let $\Delta B_n^{**} = \sup_{\tau_{n-1}\le t\le\tau_n}(B(\tau_{n-1}) - B(t))$*. If* $\mathsf{E}[\exp\{Q(\Delta B_n^{**} + b\Delta\tau_n)\}] < \infty$ *(*$n = 0, 1$*) for some* $Q \in \mathcal{SC}$*, then for all* $x, u \geq 0$*,*

$$\mathsf{P}\left(\inf_{0\le t\le x}\{B(t) - bt\} < -u\right) \le C\left(e^{-cu^2/x} + e^{-cx} + xe^{-cQ(u)}\right).$$

In the two above inequalities, C *and* c *are independent of* x *and* u*.*

Remark 2.3 Lemma 2.1 (i) is a slight extension of Proposition 1 in [25], though the latter assumes that $\Delta B_1 \geq 0$ w.p.1.

2.2 Dominatedly varying distributions

The following is the definition of the dominated variation class.

Definition 2.3 A nonnegative r.v. X and its d.f. F_X belong to the dominated variation class $\mathcal{D}$ if $\overline{F}_X(x) > 0$ for all $x \geq 0$ and

$$\limsup_{x\to\infty} \frac{\overline{F}_X(vx)}{\overline{F}_X(x)} < \infty,$$

for some (thus for all) $v \in (0, 1)$.

Remark 2.4 It should be noted (see, e.g., Section 1.4 in [13], and [11, 12]) that

$$\mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{S}, \;\; \mathcal{D} \not\subset \mathcal{S}, \;\; \mathcal{S} \not\subset \mathcal{D}.$$

For any d.f. F, let

$$\overline{F}_*(v) = \liminf_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)}, \quad \overline{F}^*(v) = \limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)}, \quad v > 0,$$

and let

$$J_F^+ = -\lim_{v\to\infty} \frac{\log \overline{F}_*(v)}{\log v}, \quad J_F^- = -\lim_{v\to\infty} \frac{\log \overline{F}^*(v)}{\log v}.$$

Strictly, J_F^+ and J_F^- are called the upper and lower Matuszewska indices of the function $1/\overline{F}(x)$ on $[0,\infty)$ (see, e.g., Section 2.1 in [6]). For simplicity, however, they are sometimes called the upper and lower Matuszewska indices of d.f. F.

Proposition 2.1 (Proposition 2.2.1 in [6]) *Suppose $F \in \mathcal{D}$. Then for any $\alpha_1 < J_F^-$ and $\alpha_2 > J_F^+$, there exist positive numbers $x_i > 0$ and $C_i > 0$ ($i = 1,2$) such that*

$$\begin{aligned} \frac{\overline{F}(x)}{\overline{F}(y)} &\le C_1 \left(\frac{x}{y}\right)^{-\alpha_1}, & \forall x \ge \forall y \ge x_1, \\ \frac{\overline{F}(x)}{\overline{F}(y)} &\ge C_2 \left(\frac{x}{y}\right)^{-\alpha_2}, & \forall x \ge \forall y \ge x_2. \end{aligned}$$

Therefore, $x^{-\alpha} = o(\overline{F}(x))$ for all $\alpha > J_F^+$.

3 Main Results

This section consists of three subsections where we present several theorems on the asymptotic equality (1.2) or (1.6). Subsection 3.1 extends the existing results to the general cumulative process described in Section 1. However such a generalization is not completely performed in the case where T is consistently varying and independent of $\{B(t)\}$. To solve this problem, subsection 3.2 discusses a special case where $B(t) = B(\lfloor t \rfloor)$ for all $t \ge 0$ and $\{B(n); n = 0,1,\dots\}$ is the additive component of a discrete-time Markov additive process. Subsection 3.3 extends the results in subsection 3.2 to a continuous-time Markov additive process.

3.1 General case: cumulative process

In this subsection, we assume $b = 1$, i.e., $\mathsf{E}[\Delta B_1] = \mathsf{E}[\Delta\tau_1]$ without loss of generality. Indeed, let $\widehat{B}(t) = B(t)/b$ for $t \ge 0$. It then follows that $\{\widehat{B}(t)\}$ is a cumulative process with the same regenerative points as those of $\{B(t)\}$, and that the asymptotic equality (1.2) is equivalent to $\mathsf{P}(\widehat{B}(T) > x) \overset{x}{\sim} \mathsf{P}(\widehat{M}(T) > x) \overset{x}{\sim} \mathsf{P}(T > x)$, where $\widehat{M}(t) = \sup_{0\le u\le t} \widehat{B}(t)$ for $t \ge 0$.

3.1.1 Non-independent-sampling case

Theorem 3.1 *Suppose* $\{B(t); t \ge 0\}$ *is nondecreasing with* t. *Further, suppose (i)* $T \in \mathcal{L}^{1/\theta}$ *for some* $0 < \theta \le 1/3$*; and (ii)* $\mathsf{E}[\exp\{Q(\Delta\tau_n)\}] < \infty$ *and* $\mathsf{E}[\exp\{Q(\Delta B_n)\}] < \infty$ *(*$n = 0, 1$*) for some* $Q \in \mathcal{SC}$ *such that*

$$x^{3\theta/2} = O(Q(x)). \tag{3.1}$$

Then $\mathsf{P}(B(T) > x) \overset{x}{\sim} \mathsf{P}(T > x)$.

Remark 3.1 If $\{B(t); t \ge 0\}$ is nondecreasing with t, then $M(t) = B(t)$ for all $t \ge 0$.

Remark 3.2 In order to prove Theorem 3.1, we use Lemma 2.1, which requires $\mathsf{E}[(\Delta\tau_1)^2] < \infty$ and $\mathsf{E}[(\Delta B_1)^2] < \infty$ though they are not explicitly assumed. These two conditions are satisfied due to condition (ii) of Theorem 3.1 and the nonnegativity of $\Delta\tau_1$ and ΔB_1. Indeed, if a nonnegative r.v. X satisfies $\mathsf{E}[e^{Q(X)}] < \infty$ for some cumulative hazard function $Q \in \mathcal{SC}$, then $\mathsf{E}[X^p] < \infty$ for any $p \ge 0$ because $e^{Q(x)} \ge x^p$ for sufficiently large $x > 0$ (see condition (ii) of Definition 2.2).

We make a comparison of Theorem 3.1 and Proposition 1.1. For this purpose, suppose $\{B(t)\}$ satisfies (1.3). We then have $\tau_0 = 0$, $\Delta\tau_n = 1$ ($n = 1, 2, \dots$), $\Delta B_0 = 0$ and $\Delta B_n = X_n$ ($n = 1, 2, \dots$). Thus, the conditions of Theorem 3.1 are reduced to the following:

(I) $T \in \mathcal{L}^{1/\theta}$ for some $0 < \theta \le 1/3$; and

(II) $\mathsf{E}[\exp\{Q(X)\}] < \infty$ for some $Q \in \mathcal{SC}$ satisfying (3.1).

Condition (I) is equivalent to $T \in \mathcal{L}^3$. On the other hand, condition (i) of Proposition 1.1 implies that T belongs to the maximum domain of attraction of the Gumbel distribution (see, e.g., Theorem 3.3.27 in [13]). It further follows from (1.4) and $x^{2/3} = o(a(x))$ that

$$1 \ge \lim_{x\to\infty} \frac{\mathsf{P}(T > x + x^{2/3})}{\mathsf{P}(T > x)} \ge \lim_{x\to\infty} \frac{\mathsf{P}(T > x + \varepsilon a(x))}{\mathsf{P}(T > x)} = e^{-\varepsilon} \to 1, \qquad \text{as } \varepsilon \to 0,$$

which implies that $T \in \mathcal{L}^3$. Thus, condition (I) is necessary for condition (i) of Proposition 1.1. In addition, condition (II) is weaker than condition (ii) of Proposition 1.1 due to $Q(x) = o(x)$ (see Definition 2.2). As a result, we can see that the conditions of Theorem 3.1 are weaker than those of Proposition 1.1 even if $\{B(t)\}$ satisfies (1.3), i.e., $B(T)$ is reduced to the random sum of nonnegative i.i.d. r.v.s.

Theorem 3.2 *Suppose* $\{B(t); t \ge 0\}$ *is nondecreasing with* t. *Further, suppose (i)* $T \in \mathcal{C}$*; (ii)* $\mathsf{E}[(\Delta\tau_1)^2] < \infty$*; (iii)* $\mathsf{P}(\Delta\tau_n > x) = o(\mathsf{P}(T > x))$ *and* $\mathsf{P}(\Delta B_n > x) = o(\mathsf{P}(T > x))$ *(*$n = 0, 1$*); (iv)* $x\mathsf{P}(|\Delta B_1 - \Delta\tau_1| > x) = o(\mathsf{P}(T > x))$*; and (v) either of the following is satisfied:*

(a) $\mathsf{E}[|\Delta B_1 - \Delta\tau_1|^r] < \infty$ *for some* $r > 1$*; or*

(b) $\int_y^\infty x^{-1}\mathsf{P}(T > x)\mathrm{d}x < \infty$ *for some* $y \in (0, \infty)$.

Then $\mathsf{P}(B(T) > x) \overset{x}{\sim} \mathsf{P}(T > x)$.

Remark 3.3 We prove Theorem 3.2 by using Lemma A.8, which requires condition (ii).

Remark 3.4 In Theorem 3.2, the asymptotic upper and lower bounds, $B(T) \lesssim_x \mathsf{P}(T > x)$ and $B(T) \gtrsim_x \mathsf{P}(T > x)$, are proved by the same approach with the large deviation bounds for the maxima of partial sums of i.i.d. r.v.s (Lemmas A.9 and A.10). However, the conditions required by the two asymptotic bounds are slightly different. The difference appears in conditions (iv) and (v): the upper one requires that $x\mathsf{P}(\Delta B_1 - \Delta\tau_1 > x) = o(\mathsf{P}(T > x))$ and either of the following holds:

(a) $\mathsf{E}[\{(\Delta B_1 - \Delta\tau_1)^+\}^r] < \infty$ for some $r > 1$ or

(b) $\int_y^\infty x^{-1}\mathsf{P}(T > x)\mathrm{d}x < \infty$ for some $y \in (0, \infty)$;

whereas the lower one requires that $x\mathsf{P}(\Delta\tau_1 - \Delta B_1 > x) = o(\mathsf{P}(T > x))$ and either of the following holds:

(a) $\mathsf{E}[\{(\Delta\tau_1 - \Delta B_1)^+\}^r] < \infty$ for some $r > 1$ or

(b) $\int_y^\infty x^{-1}\mathsf{P}(T > x)\mathrm{d}x < \infty$ for some $y \in (0, \infty)$.

It should be noted that these conditions are integrated into conditions (iv) and (v).

We now compare Theorem 3.2 with Proposition 1.2, assuming that $\{B(t)\}$ satisfies (1.3). Under this assumption, the conditions of Theorem 3.2 are reduced to the following:

(I) $T \in \mathcal{C}$;

(II) $x\mathsf{P}(X > x) = o(\mathsf{P}(T > x))$; and

(III) either of the following is satisfied:

  (A) $\mathsf{E}[X^r] < \infty$ for some $r > 1$; or

  (B) $\int_y^\infty x^{-1}\mathsf{P}(T > x)\mathrm{d}x < \infty$ for some $y \in (0, \infty)$.

Clearly, the set of conditions (I), (II) and (III.A) is the same as that of conditions (i), (ii) and (iii) of Proposition 1.2. Further, the set of conditions (I), (II) and (III.B) does not imply that of conditions (I), (II) and (III.A). In fact, suppose $\mathsf{P}(T > x) \overset{x}{\sim} (\log x)^{-2}$ and $\mathsf{P}(X >$

$x) \stackrel{x}{\sim} x^{-1}(\log x)^{-3}$. We then have $T \in \mathcal{C}$ and $x\mathsf{P}(X > x) = o(\mathsf{P}(T > x))$. It also holds that $\mathsf{E}[X] < \infty$ and $\int_y^\infty x^{-1}\mathsf{P}(T > x)\mathrm{d}x < \infty$ for some $y \in (0, \infty)$, which follow from

$$\int_y^\infty \frac{\mathrm{d}x}{x(\log x)^m} = \frac{1}{(m-1)(\log y)^{m-1}}, \quad y > 1,\ m \neq 1.$$

Thus conditions (I), (II) and (III.B) are satisfied. However, condition (III.A) does not hold, i.e., $\mathsf{E}[X^r] = \infty$ for any $r > 1$ because

$$\mathsf{P}(X^r > x) \stackrel{x}{\sim} \frac{r^3}{x^{1/r}(\log x)^3} \gtrsim_x \frac{r^3}{x^{(1/r)+(r-1)/(2r)}} = \frac{r^3}{x^{(r+1)/(2r)}},$$

where $0 < (r+1)/(2r) < 1$ for $r > 1$.

Consequently, the conditions of Theorem 3.2 are still weaker than those of Proposition 1.2 in the context of the random sum of nonnegative i.i.d. r.v.s.

3.1.2 Independent-sampling case

Theorem 3.3 *Suppose T is independent of $\{B(t); t \ge 0\}$. Further, suppose (i) $T \in \mathcal{L}^{1/\theta}$ for some $0 < \theta \le 1/2$; (ii) $\mathsf{E}[(\Delta\tau_1)^2] < \infty$ and $\mathsf{E}[(\Delta B_1)^2] < \infty$; and (iii) $\mathsf{E}[\exp\{Q(\Delta B_n^*)\}] < \infty$ ($n = 0, 1$) for some $Q \in \mathcal{SC}$ such that $x^\theta = O(Q(x))$. Then $\mathsf{P}(B(T) > x) \stackrel{x}{\sim} \mathsf{P}(M(T) > x) \stackrel{x}{\sim} \mathsf{P}(T > x)$.*

Remark 3.5 We use Lemma 2.1 (i) to prove $\mathsf{P}(M(T) > x) \lesssim_x \mathsf{P}(T > x)$. For this purpose, conditions (ii) and (iii) are assumed. Further, the proof of $\mathsf{P}(B(T) > x) \gtrsim_x \mathsf{P}(T > x)$ requires the central limit theorem (CLT) for $\{B(t)\}$, which holds under condition (ii) (see, e.g., [3, Chapter VI, Theorem 3.2]).

Note that condition (i) of Theorem 3.3 is equivalent to condition (i) of Proposition 1.3. Note also that condition (ii) of Theorem 3.3 is weaker than the corresponding condition of Proposition 1.3 because the positivity of ΔB_n and condition (iii) of Proposition 1.3 imply $\mathsf{E}[(\Delta B_1)^2] < \infty$ (see Remark 3.2). In addition, if $Q(x) = \eta\sqrt{x}$ for some $\eta > 0$, then condition (iii) of Theorem 3.3 is reduced to condition (iii) of Proposition 1.3. Consequently, Theorem 3.3 is a more general result than Proposition 1.3.

Theorem 3.4 *Suppose T is independent of $\{B(t); t \ge 0\}$. Further, suppose (i) $T \in \mathcal{C}$; (ii) $\mathsf{E}[\sup_{\tau_0 \le t \le \tau_1} |B(t) - B(\tau_0)|] < \infty$ and $\mathsf{E}[(\Delta\tau_1)^2] < \infty$; (iii) $\mathsf{P}(\Delta B_n^* > x) = o(\mathsf{P}(T > x))$ ($n = 0, 1$); (iv) $x\mathsf{P}(\Delta B_1 - \Delta\tau_1 > x) = o(\mathsf{P}(T > x))$; and (v) either of the following is satisfied:*

(a) $\mathsf{E}[\{(\Delta B_1 - \Delta\tau_1)^+\}^r] < \infty$ *for some* $r > 1$*; or*

(b) $\int_y^\infty x^{-1}\mathsf{P}(T > x)\mathrm{d}x < \infty$ *for some* $y \in (0, \infty)$.

Then $\mathsf{P}(B(T) > x) \overset{x}{\sim} \mathsf{P}(M(T) > x) \overset{x}{\sim} \mathsf{P}(T > x)$.

Remark 3.6 The asymptotic upper bound $\mathsf{P}(M(T) > x) \lesssim_x \mathsf{P}(T > x)$ is proved by almost the same approach as Theorem 3.2, and its proof requires the same conditions except for the monotonicity of $B(t)$. On the other hand, the asymptotic lower bound $\mathsf{P}(B(T) > x) \gtrsim_x \mathsf{P}(T > x)$ is proved by a different approach using the SLLN for $\{B(t)\}$, which requires $\mathsf{E}[\sup_{\tau_0 \le t \le \tau_1} |B(t) - B(\tau_0)|] < \infty$ in condition (ii) (see [3, Chapter VI, Theorem 3.1]).

Remark 3.7 Suppose $\{B(t)\}$ is nondecreasing with t. Then $\Delta B_1 = \Delta B_1^* = \sup_{\tau_0 \le t \le \tau_1} |B(t) - B(\tau_0)|$, which implies that $\mathsf{E}[\sup_{\tau_0 \le t \le \tau_1} |B(t) - B(\tau_0)|] < \infty$ is satisfied by our basic assumption (1.1). Thus conditions (i)–(v) of Theorem 3.2 are sufficient for conditions (i)–(v) of Theorem 3.4.

We make a comparison of Theorem 3.4 with Propositions 1.4 and 1.5. Suppose that $\{B(t)\}$ satisfies (1.3). It then follows that the conditions of Theorem 3.4 are reduced to the following:

(I) $T \in \mathcal{C}$;

(II) $x\mathsf{P}(X > x) = o(\mathsf{P}(T > x))$; and

(III) $\mathsf{E}[X^r] < \infty$ for some $r > 1$ or $\int_y^\infty x^{-1}\mathsf{P}(T > x)\mathrm{d}x < \infty$ for some $y \in (0, \infty)$.

Theorem 3.4 does not necessarily require either the condition $\mathsf{E}[T] < \infty$ of Proposition 1.4 or condition (ii) of Proposition 1.5, whereas condition (II) is not necessary for Propositions 1.4 and 1.5 (see Remark below Theorem 3.2 in [37]). As a result, Theorem 3.4 is not a complete generalization of Propositions 1.4 and 1.5.

3.2 Special case: discrete-time Markov additive process

In this subsection, we extend Propositions 1.4 and 1.5 to the random sum of (possibly negative) r.v.s with Markovian correlation. For this purpose, we introduce a discrete-time Markov additive process.

Let $\{J_n; n = 0, 1, \dots\}$ is a discrete-time Markov chain with a finite state space $\mathbb{D} := \{0, 1, \dots, d-1\}$. Let X_n's ($n = 0, 1, \dots$) denote r.v.s such that

$$\mathsf{P}(X_0 \le x, J_0 = j) = \beta_i(x), \quad x \in \mathbb{R},$$

and for $n = 0, 1, \dots,$

$$\mathsf{P}(X_{n+1} \le x, J_{n+1} = j \mid J_n = i) = H_{i,j}(x), \quad x \in \mathbb{R}.$$

Further, let $S_n = \sum_{i=0}^n X_i$ for $n = 0, 1, \dots$. It then follows that $\{(S_n, J_n); n = 0, 1, \dots\}$ is a Markov additive process with initial distribution $\boldsymbol{\beta}(x) = (\beta_i(x))_{i \in \mathbb{D}}$ and Markov additive kernel

(called "kernel" for short) $\boldsymbol{H}(x) = (H_{i,j}(x))_{i,j\in\mathbb{D}}$ ($x \in \mathbb{R}$). For later use, let $\widehat{\boldsymbol{\beta}}(\xi)$ and $\widehat{\boldsymbol{H}}(\xi)$ denote the characteristic functions of $\boldsymbol{\beta}(x)$ and $\boldsymbol{H}(x)$, i.e.,

$$\widehat{\boldsymbol{\beta}}(\xi) = \int_{x\in\mathbb{R}} e^{\mathrm{i}\xi x}\mathrm{d}\boldsymbol{\beta}(x), \qquad \widehat{\boldsymbol{H}}(\xi) = \int_{x\in\mathbb{R}} e^{\mathrm{i}\xi x}\mathrm{d}\boldsymbol{H}(x),$$

respectively, where $\mathrm{i} = \sqrt{-1}$. For simplicity, we omit "(0)" of vector $\widehat{\boldsymbol{\beta}}(0)$, matrix $\widehat{\boldsymbol{H}}(0)$ and their elements.

In what follows, we make the following assumption:

Assumption 3.1 (i) Let $B(t) = S_{\lfloor t\rfloor} = \sum_{n=0}^{\lfloor t\rfloor} X_n$ for $t \ge 0$;

(ii) the background process $\{J_n\}$ is irreducible, i.e., $\widehat{\boldsymbol{H}} = \boldsymbol{H}(\infty)$ is an irreducible stochastic matrix; and

(iii) the mean drift of the additive component $\{S_n\}$ is finite and positive, i.e.,

$$h := \boldsymbol{\varpi}\int_{x\in\mathbb{R}} x\mathrm{d}\boldsymbol{H}(x)\boldsymbol{e} \in (0,\infty), \tag{3.2}$$

where $\boldsymbol{\varpi} = (\varpi_i)_{i\in\mathbb{D}}$ is the stationary probability vector of $\boldsymbol{H}(\infty)$, and where $\boldsymbol{e}$ is a column vector of ones with an appropriate dimension.

It is easy to see that $\{B(t); t \ge 0\}$ is a cumulative process because $\{(B(n), J_n); n = 0,1,\dots\}$ is a discrete-time Markov additive process. Let $0 \le \tau_0 < \tau_1 < \cdots$ denote hitting times of $\{J_n\}$ to state zero, which are regenerative points of $\{B(t)\}$. Let $\widehat{\psi}_0(z,\xi) = \mathsf{E}[z^{\Delta\tau_0}e^{\mathrm{i}\xi\Delta B_0}]$ and $\widehat{\psi}_1(z,\xi) = \mathsf{E}[z^{\Delta\tau_1}e^{\mathrm{i}\xi\Delta B_1}]$. We then have

$$\widehat{\psi}_0(z,\xi) = z\widehat{\beta}_0(\xi) + z\widehat{\boldsymbol{\beta}}_+(\xi)\left(\boldsymbol{I} - z\widehat{\boldsymbol{H}}_+(\xi)\right)^{-1} z\widehat{\boldsymbol{h}}_+(\xi), \tag{3.3}$$

$$\widehat{\psi}_1(z,\xi) = z\widehat{H}_{0,0}(\xi) + z\widehat{\boldsymbol{\eta}}_+(\xi)\left(\boldsymbol{I} - z\widehat{\boldsymbol{H}}_+(\xi)\right)^{-1} z\widehat{\boldsymbol{h}}_+(\xi), \tag{3.4}$$

where $\boldsymbol{I}$ denotes the identity matrix with an appropriate dimension and

$$\widehat{\boldsymbol{\beta}}(\xi) = \begin{array}{c} \begin{array}{cc} \{0\} & \mathbb{D}\setminus\{0\} \end{array} \\ \begin{pmatrix} \widehat{\beta}_0(\xi) & \widehat{\boldsymbol{\beta}}_+(\xi) \end{pmatrix} \end{array}, \quad \widehat{\boldsymbol{H}}(\xi) = \begin{array}{cc} & \begin{array}{cc} \{0\} & \mathbb{D}\setminus\{0\} \end{array} \\ \begin{array}{c} \{0\} \\ \mathbb{D}\setminus\{0\} \end{array} & \begin{pmatrix} \widehat{H}_{0,0}(\xi) & \widehat{\boldsymbol{\eta}}_+(\xi) \\ \widehat{\boldsymbol{h}}_+(\xi) & \widehat{\boldsymbol{H}}_+(\xi) \end{pmatrix} \end{array}.$$

The first term of (3.4) corresponds to the event where a regenerative cycle lasts only for a unit of time, i.e., the background process $\{J_n\}$ moves from state zero to state zero in one transition. The second term corresponds to the event where $\{J_n\}$ moves from state zero to a state in $\mathbb{D}\setminus\{0\}$ and then eventually returns to state zero. The equation (3.3) can be understood in a similar way.

Fixing $\xi = 0$ in (3.3) and (3.4) and taking the inverse of them with respect to z, we have

$$\begin{aligned} \mathsf{P}(\Delta\tau_0 = k) &= \mathbb{1}(k=1)\widehat{\beta}_0 + \mathbb{1}(k \ge 2)\widehat{\boldsymbol{\beta}}_+(\widehat{\boldsymbol{H}}_+)^{k-2}\widehat{\boldsymbol{h}}_+, \qquad & k = 0,1,2,\dots, \\ \mathsf{P}(\Delta\tau_1 = k) &= \mathbb{1}(k=1)\widehat{H}_{0,0} + \mathbb{1}(k \ge 2)\widehat{\boldsymbol{\eta}}_+(\widehat{\boldsymbol{H}}_+)^{k-2}\widehat{\boldsymbol{h}}_+, \qquad & k = 0,1,2,\dots. \end{aligned} \tag{3.5}$$

Therefore, $\Delta\tau_0$ and $\Delta\tau_1$ follow discrete phase-type distributions [29]. Further, we have the following result by using the renewal reward theory (see, e.g., [45, Chapter 2, Theorem 2]) and the discrete-time version of the ergodic theorem (see, e.g., [7, Chapter 3, Theorem 4.1]):

Proposition 3.1 *Under Assumption 3.1,*

$$b := \frac{\mathsf{E}[\Delta B_1]}{\mathsf{E}[\Delta\tau_1]} = \boldsymbol{\varpi} \int_{x \in \mathbb{R}} x \mathrm{d}\boldsymbol{H}(x)\boldsymbol{e} = h \in (0,\infty).$$

In what follows, we present two theorems that supersede Propositions 1.4 and 1.5. Before doing this, we introduce three lemmas for the proofs of the theorems.

Lemma 3.1 *Suppose Assumptions 3.1 holds. Further, let $\overline{\boldsymbol{\beta}}(x) = \int_x^\infty \mathrm{d}\boldsymbol{\beta}(y)$ and $\overline{\boldsymbol{H}}(x) = \int_x^\infty \mathrm{d}\boldsymbol{H}(y)$ for $x \in \mathbb{R}$ and suppose there exist some $c^* \in [0,\infty)$ and some nonnegative r.v. $Y \in \mathcal{S}$ such that*

$$\limsup_{x\to\infty} \frac{\overline{\boldsymbol{\beta}}(x)}{\mathsf{P}(Y > x)} \le c^* \widetilde{\boldsymbol{\beta}}, \qquad \limsup_{x\to\infty} \frac{\overline{\boldsymbol{H}}(x)}{\mathsf{P}(Y > x)} \le c^* \widetilde{\boldsymbol{H}},$$

where $\widetilde{\boldsymbol{\beta}} = (\widetilde{\beta}_i)_{i\in\mathbb{D}}$ is a finite nonnegative vector and $\widetilde{\boldsymbol{H}} = (\widetilde{H}_{i,j})_{i,j\in\mathbb{D}}$ is a finite nonnegative matrix. We then have

$$\mathsf{P}(\Delta B_n > x) \lesssim_x c^* C \mathsf{P}(Y > x), \quad n = 0,1.$$

Lemma 3.2 *If the assumptions of Lemma 3.1 are satisfied, then*

$$\limsup_{x\to\infty} \frac{\mathsf{P}(\Delta B_1 > x \mid \Delta\tau_1 = k)}{\mathsf{P}(Y > x)} \le c^* C k, \qquad \textit{for all } k = 1,2,\dots,$$

where C is independent of k.

Lemma 3.3 *If the assumptions of Lemma 3.1 are satisfied, then for all $t \ge 0$ and $m = 0,1,\dots,$*

$$\mathsf{P}\left(\sum_{i=1}^m \Delta B_i > x \,\middle|\, N(t) = m \right) \lesssim_x c^* C t \mathsf{P}(Y > x),$$

where $N(t) = \max\{k \ge 0; \sum_{i=1}^k \Delta\tau_i \le t\}$ for $t \ge 0$.

The following theorems present two sets of conditions for (1.2). Note here that under Assumption 3.1, the asymptotic equality (1.2) is reduced to $\mathsf{P}(S_{\lfloor T \rfloor} > hx) \stackrel{x}{\sim} \mathsf{P}(M_{\lfloor T \rfloor} > hx) \stackrel{x}{\sim} \mathsf{P}(T > x)$, where $M_n = \max_{0\le k\le n} S_k$.

Theorem 3.5 *Suppose Assumption 3.1 holds and T is independent of the Markov additive process $\{(S_n, J_n)\}$. Further, suppose $T \in \mathcal{C}$, $\mathsf{E}[T] < \infty$ and*

$$\int_{|y|>x} \mathrm{d}\boldsymbol{\beta}(y) = o(\mathsf{P}(T > x)), \qquad \int_{|y|>x} \mathrm{d}\boldsymbol{H}(y) = o(\mathsf{P}(T > x)). \tag{3.6}$$

Then $\mathsf{P}(S_{\lfloor T \rfloor} > hx) \stackrel{x}{\sim} \mathsf{P}(M_{\lfloor T \rfloor} > hx) \stackrel{x}{\sim} \mathsf{P}(T > x)$.

Theorem 3.6 *Suppose Assumption 3.1 holds and T is independent of the Markov additive process $\{(S_n, J_n)\}$. Further, suppose $T \in \mathcal{C}$ and there exists some nonnegative r.v. $Y \in \mathcal{S}$ such that*

$$\int_{|y|>x} \mathrm{d}\boldsymbol{\beta}(y) = O(\mathsf{P}(Y > x)), \qquad \int_{|y|>x} \mathrm{d}\boldsymbol{H}(y) = O(\mathsf{P}(Y > x)), \tag{3.7}$$

$$\lim_{x\to\infty} \mathsf{E}[T \cdot \mathbb{1}(T \le x, N(T) \le x/\mathsf{E}[\Delta\tau_1])] \frac{\mathsf{P}(Y > x)}{\mathsf{P}(T > x)} = 0. \tag{3.8}$$

Then $\mathsf{P}(S_{\lfloor T \rfloor} > hx) \stackrel{x}{\sim} \mathsf{P}(M_{\lfloor T \rfloor} > hx) \stackrel{x}{\sim} \mathsf{P}(T > x)$.

Remark 3.8 The equation (3.8) holds if

$$\lim_{x\to\infty} \mathsf{E}[T \cdot \mathbb{1}(T \le x)] \frac{\mathsf{P}(Y > x)}{\mathsf{P}(T > x)} = 0.$$

It is easy to see that Theorems 3.5 and 3.6 are more general than Propositions 1.4 and 1.5, respectively, even if $B(T)$ is reduced to the random sum of i.i.d. r.v.s.

3.3 Special case: continuous-time Markov additive process

In this subsection, we consider a continuous-time Markov additive process $\{(B(t), J(t)); t \ge 0\}$ with state space $\mathbb{R} \times \mathbb{D}$ and kernel $\boldsymbol{D}(x) = (D_{i,j}(x))_{i,j\in\mathbb{D}}$, where $\{B(t)\}$ is the additive component and $\{J(t)\}$ is the background process. For simplicity, we assume $B(0) = 0$. We then have for all $t \ge 0$,

$$\mathsf{E}[\exp\{\mathrm{i}\xi B(t)\} \cdot \mathbb{1}(J(t) = j) \mid J(0) = i] = \left[\exp\{\widehat{\boldsymbol{D}}(\xi)t\}\right]_{i,j}, \quad i, j \in \mathbb{D}, \tag{3.9}$$

where $\widehat{\boldsymbol{D}}(\xi) = \int_{x\in\mathbb{R}} e^{\mathrm{i}\xi x}\mathrm{d}\boldsymbol{D}(x)$ and $[\,\cdot\,]_{i,j}$ denotes the (i, j)th element of the matrix between square brackets.

In what follows, we make the following assumption:

Assumption 3.2 (i) The equation (3.9) holds for all $t \ge 0$; (ii) $\widehat{\boldsymbol{D}}(0) = \boldsymbol{D}(\infty)$ is an irreducible stochastic matrix; and (iii) $\boldsymbol{\pi} \int_{x\in\mathbb{R}} x\mathrm{d}\boldsymbol{D}(x)\boldsymbol{e} \in (0, \infty)$, where $\boldsymbol{\pi} = (\pi_i)_{i\in\mathbb{D}}$ denotes the stationary probability vector of $\widehat{\boldsymbol{D}}(0)$.

Under Assumption 3.2, $\{B(t)\}$ is a cumulative process, and it follows from the renewal reward theory (see, e.g., [45, Chapter 2, Theorem 2]) and the continuous-time version of the ergodic theorem (see, e.g., [7, Chapter 8, Theorem 6.2]) that

$$b := \frac{\mathsf{E}[\Delta B_1]}{\mathsf{E}[\Delta \tau_1]} = \boldsymbol{\pi} \int_{x\in\mathbb{R}} x \mathrm{d}\boldsymbol{D}(x)\boldsymbol{e} \in (0,\infty).$$

Further, it follows from (3.9) that

$$\begin{aligned}
&\mathsf{E}[\exp\{\mathrm{i}\xi B(T)\} \cdot \mathbb{1}(J(t) = j) \mid J(0) = i] \\
&\quad = \left[\int_0^\infty \exp\{\widehat{\boldsymbol{D}}(\xi)t\}\mathrm{d}\mathsf{P}(T \le t)\right]_{i,j} \\
&\quad = \sum_{n=0}^\infty \int_0^\infty e^{-\gamma t}\frac{(\gamma t)^n}{n!}\mathrm{d}\mathsf{P}(T \le t) \cdot \left[\left\{\boldsymbol{I} + \gamma^{-1}\widehat{\boldsymbol{D}}(\xi)\right\}^n\right]_{i,j} \\
&\quad =: \sum_{n=0}^\infty p_n \cdot \left[\{\widehat{\boldsymbol{K}}(\xi)\}^n\right]_{i,j},
\end{aligned} \tag{3.10}$$

where $\gamma = \max_{i\in\mathbb{D}} |D_{i,i}(\infty)|$. Note here that $\boldsymbol{K}(x) := \boldsymbol{I} + \gamma^{-1}\boldsymbol{D}(x)$ ($x \in \mathbb{R}$) can be considered as the kernel of a discrete-time Markov additive process $\{(S_n, J_n)\}$ discussed in the previous subsection. Note also that $\{p_n; n = 0, 1, \dots\}$ is the distribution of the counting process of Poisson arrivals with rate γ during time interval $(0, T]$. It is easy to see that if $T \in \mathcal{C}$, then such a counting process is a cumulative process that satisfies all the conditions of Theorem 3.4 and thus

$$\sum_{n=k+1}^\infty p_n \overset{k}{\sim} \mathsf{P}(T > k/\gamma).$$

The equation (3.10) implies that $(B(T), J(T))$ is equivalent to a discrete-time Markov additive process $\{(S_n, J_n)\}$ with initial condition $S_0 = 0$ (i.e., $\boldsymbol{\beta}(x)\boldsymbol{e} = 0$ for all $x > 0$) and kernel $\boldsymbol{H}(x) = \boldsymbol{K}(x)$ ($x \in \mathbb{R}$) sampled at random time T', where T' denotes a nonnegative r.v. such that $\mathsf{P}(T' = n) = p_n$ ($n = 0, 1, \dots$) and T' is independent of $\{(S_n, J_n)\}$. As a result, using Theorems 3.5 and 3.6, we can readily prove the following corollaries, whose proofs are omitted.

Corollary 3.1 *Suppose Assumption 3.2 holds and T is independent of the Markov additive process $\{(B(t), J(t))\}$. Further, suppose $T \in \mathcal{C}$, $\mathsf{E}[T] < \infty$ and*

$$\int_{|y|>x} \mathrm{d}\boldsymbol{D}(y) = o(\mathsf{P}(T > x)).$$

Then $\mathsf{P}(B(T) > bx) \overset{x}{\sim} \mathsf{P}(M(T) > bx) \overset{x}{\sim} \mathsf{P}(T > x)$.

Corollary 3.2 *Suppose Assumption 3.2 holds and T is independent of the Markov additive process $\{(B(t), J(t))\}$. Further, suppose $T \in \mathcal{C}$ and there exists some nonnegative r.v. $Y \in \mathcal{S}$ such that*

$$\int_{|y|>x} \mathrm{d}\boldsymbol{D}(y) = O(\mathsf{P}(Y > x)), \qquad \lim_{x\to\infty} \mathsf{E}[T \cdot 1\!\!1(T \le x)] \frac{\mathsf{P}(Y > x)}{\mathsf{P}(T > x)} = 0.$$

Then $\mathsf{P}(B(T) > bx) \overset{x}{\sim} \mathsf{P}(M(T) > bx) \overset{x}{\sim} \mathsf{P}(T > x)$.

4 Application

In this section, we first introduce a new (discrete-time) on/off arrival process, ON/OFF-BMAP, mentioned in Section 1. We then apply our main results to the subexponential asymptotics of the loss probability of a single-server finite-buffer queue with an ON/OFF-BMAP and deterministic service times.

4.1 ON/OFF batch Markovian arrival process

We describe the definition of ON/OFF-BMAPs in discrete time. The time interval $[n, n+1]$ ($n \in \mathbb{Z}$) is called slot n, where $\mathbb{Z} = \{0, \pm 1, \pm 2 \dots\}$. The ON/OFF-BMAP is an on/off arrival process, where on-periods and off-periods are repeated alternately. The lengths of off-periods are i.i.d., and no arrivals occur in off-periods. On the other hand, during an on-period, at least one arrival occurs in each slot w.p.1 and the numbers of arrivals in respective slots follow a BMAP started with some initial distribution at the beginning of the on-period. Further, the lengths of on-periods are i.i.d., but the length of each on-period may depend on the BMAP in the on-period. For simplicity, the BMAP in the mth ($m \in \mathbb{Z}$) on-period is called the mth BMAP.

To describe the ON/OFF-BMAP more precisely, we define some notations. Let $N_{m,n}$ ($m \in \mathbb{Z}$, $n = 0, 1, \dots$) denote the number of arrivals in the nth slot of the mth on-period. Let $J_{m,0}, J_{m,1}, J_{m,2}, \dots$ ($m \in \mathbb{Z}$) denote the background states of the mth BMAP. We then assume that

$$\mathsf{P}(N_{m,0} = k, J_{m,0} = i) = \alpha_i(k), \qquad i \in \mathbb{D},\ k = 1, 2, \dots,$$

where $\boldsymbol{\alpha}(k) = (\alpha_i(k))_{i\in\mathbb{D}}$ is a $1 \times d$ nonnegative vector such that $\widehat{\boldsymbol{\alpha}} := \sum_{k=1}^{\infty} \boldsymbol{\alpha}(k)$ is a probability vector. We also assume that for $n = 1, 2, \dots,$

$$\mathsf{P}(N_{m,n} = k, J_{m,n} = j \mid J_{m,n-1} = i) = \Lambda_{i,j}(k), \qquad i, j \in \mathbb{D},\ k = 1, 2, \dots,$$

where $\boldsymbol{\Lambda}(k) = (\Lambda_{i,j}(k))_{i,j\in\mathbb{D}}$ is a $d \times d$ substochastic matrix such that $\widehat{\boldsymbol{\Lambda}} := \sum_{k=1}^{\infty} \boldsymbol{\Lambda}(k)$ is an irreducible stochastic matrix.

Let T_m ($m \in \mathbb{Z}$) denote the length of the mth on-period. Let Φ_m ($m \in \mathbb{Z}$) denote

$$\Phi_m = \{T_m, (N_{m,0}, J_{m,0}), (N_{m,1}, J_{m,1}), \dots, (N_{m,T_m-1}, J_{m,T_m-1})\}.$$

We then assume that the Φ_m's ($m \in \mathbb{Z}$) are i.i.d. Thus the T_m's ($m \in \mathbb{Z}$) are i.i.d. r.v.s though each T_m may depend on the mth BMAP, i.e., $\{(N_{m,n}, J_{m,n}); n = 0, 1, \dots\}$.

For later use, let λ denote the arrival rate during on periods. We then have

$$\lambda = \boldsymbol{\phi} \sum_{k=1}^{\infty} k \boldsymbol{\Lambda}(k) \boldsymbol{e} \ge 1, \tag{4.1}$$

where $\boldsymbol{\phi} = (\phi_i)_{i \in \mathbb{D}}$ denotes the stationary probability vector of $\widehat{\boldsymbol{\Lambda}}$.

Remark 4.1 The ON/OFF-BMAP is a generalization of the batch-on/off process introduced by Galmés and Puigjaner [17]. In the batch-on/off process, the numbers of arrivals in respective slots are i.i.d. and independent of the lengths of i.i.d. on-periods. Based on the Wiener-Hopf factorization (see, e.g., [3, Chapter VIII, Section 3]), Galmés and Puigjaner [18, 19] study the response time distribution of a single-server queue with a batch-on/off process and deterministic service times.

Remark 4.2 The ON/OFF-BMAP is similar to the PAP proposed by Alfa and Neuts [2] and Breuer and Alfa [8]. The PAP can be considered as a special case of the ON/OFF-BMAP in the sense that the lengths of on-periods and (resp. off-periods) follow a phase-type distribution. However, in the PAP, it is allowed that no arrivals occur in a slot of the *on-period*.

4.2 Loss probability of (ON/OFF-BMAP)/D/1/K queue

We begin with the description of our queueing model. Customers arrive at the system according to an ON/OFF-BMAP. The system has a single server and a buffer of finite capacity K. The service times of customers are all equal to the length of one slot. According to Kendall's notation, our queueing model is denoted by (ON/OFF-BMAP)/D/1/K.

For analytical convenience, we assume that arrivals in each slot of the on-period occur at the same time, immediately after the beginning of the slot. We also assume that departure points are located immediately before the ends of respective slots.

Under the above assumptions, we discuss the loss probability of the (ON/OFF-BMAP)/D/1/K queue. For this purpose, we observe the queue length process immediately after the ends of respective off-periods. Let $L_m^{(K)}$ ($m \in \mathbb{Z}$) denote the queue length immediately after the end of the mth off-period. Let I_m^{off} ($m \in \mathbb{Z}$) denote the length of the mth off-period. Further, let A_m ($m \in \mathbb{Z}$) denote

$$A_m = \sum_{n=0}^{T_m-1} (N_{m,n} - 1), \tag{4.2}$$

which is the increment in the queue length during the mth on-period. We then have

$$L_{m+1}^{(K)} = (\min(L_m^{(K)} + A_{m+1}, K) - I_{m+1}^{\rm off})^+,$$

where $(x)^+ = \max(x, 0)$. Note here that $\{A_m\}$, $\{I_m^{\rm off}\}$ and $\{T_m\}$ are sequences of i.i.d. r.v.s. For later use, let A, $I^{\rm off}$ and T denote generic r.v.s for $\{A_m\}$, $\{I_m^{\rm off}\}$ and $\{T_m\}$, respectively.

We now define $P_{\rm loss}^{(K)}$ as the loss probability, which is the time-average of losses. In the mth renewal cycle consisting of the mth on- and off-periods, the numbers of arrivals and losses are equal to $A_m + T_m$ and $(L_{m-1}^{(K)} + A_m - K)^+$, respectively. It then follows from the renewal reward theory (see, e.g., [45, Chapter 2, Theorem 2]) that

$$P_{\rm loss}^{(K)} = \frac{\mathsf{E}[(L_{m-1}^{(K)} + A_m - K)^+]}{\mathsf{E}[A_m + T_m]}.$$

4.3 Subexponential asymptotics of the loss probability

In this subsection, we derive some subexponential asymptotic formulas for the loss probability $P_{\rm loss}^{(K)}$. To achieve this, we use the following proposition:

Proposition 4.1 (Theorem 5 in [22]) *Suppose* $0 < \mathsf{E}[A] < \infty$ *and let* $A_{\rm e}$ *denote the equilibrium r.v. of* A*, i.e.,* $\mathsf{P}(A_{\rm e} \le x) = (1/\mathsf{E}[A]) \int_0^x \mathsf{P}(A > y) {\rm d}y$ *for* $x \ge 0$*. If* $\mathsf{E}[A] < \mathsf{E}[I^{\rm off}]$ *and* $A_{\rm e} \in \mathcal{S}$*, then*

$$P_{\rm loss}^{(K)} \stackrel{K}{\sim} \frac{\mathsf{E}[(A-K)^+]}{\mathsf{E}[A] + \mathsf{E}[T]} = \frac{\mathsf{E}[A]}{\mathsf{E}[A] + \mathsf{E}[T]} \mathsf{P}(A_{\rm e} > K).$$

To utilize Proposition 4.1, we consider a cumulative process $\{B(t); t \ge 0\}$ such that

$$B(t) = \sum_{n=0}^{\lfloor t \rfloor} (N_{m,n} - 1), \tag{4.3}$$

which is nondecreasing with t. For simplicity, let $X_n = N_{m,n} - 1$ and $J_n = J_{m,n}$ for $n = 0, 1, \dots$. Further, let $S_n = \sum_{l=0}^n X_l$ for $n = 0, 1, \dots$. It then follows that $B(t) = S_{\lfloor t \rfloor}$ for $t \ge 0$ and $\{(S_n, J_n); n = 0, 1, \dots\}$ is a Markov additive process with state space $\{0, 1, \dots\} \times \mathbb{D}$, initial distribution $\boldsymbol{\alpha}(k)$ and Markov additive kernel $\boldsymbol{\Lambda}(k+1)$ ($k = 0, 1, \dots$). Thus the stochastic process $\{B(t)\}$ in (4.3) is a cumulative process of the same type as that in subsection 3.2. As with subsection 3.2, let $0 \le \tau_0 < \tau_1 < \cdots$ denote hitting times of $\{J_n\}$ to state zero, which are regenerative points of $\{B(t)\}$. From (4.1) and Proposition 3.1, we have

$$b := \frac{\mathsf{E}[\Delta B_1]}{\mathsf{E}[\Delta \tau_1]} = \lambda - 1.$$

We now assume the following:

Assumption 4.1 $\lambda > 1$ and there exists some nonnegative r.v. $Y \in \mathcal{S}$ such that

$$\sum_{l=k+1}^{\infty} \boldsymbol{\alpha}(l) = O(\mathsf{P}(Y > k)), \qquad \sum_{l=k+1}^{\infty} \boldsymbol{\Lambda}(l) = O(\mathsf{P}(Y > k)).$$

In what follows, we show four subexponential asymptotic formulas for the loss probability by combining Proposition 4.1 with the main results in Section 3. The first two formulas are obtained from the results in the non-independent-sampling case, and the others are from those in the independent-sampling case.

Corollary 4.1 *Suppose Assumption 4.1 holds and* $\mathsf{E}[A] < \mathsf{E}[I^{\rm off}]$. *Further, suppose (i)* $T \in \mathcal{L}^{1/\theta}$ *for some* $0 < \theta \le 1/3$*; (ii)* $\mathsf{E}[T] < \infty$ *and the equilibrium r.v.* $T_{\rm e}$ *of* T *is subexponential (i.e.,* $T_{\rm e} \in \mathcal{S}$*); and (iii)* $\mathsf{E}[\exp\{Q(Y)\}] < \infty$ *for some* $Q \in \mathcal{SC}$ *such that* $x^{3\theta/2} = O(Q(x))$. *We then have*

$$P_{\rm loss}^{(K)} \stackrel{K}{\sim} \frac{(\lambda-1)\mathsf{E}[T]}{\mathsf{E}[A]+\mathsf{E}[T]} \mathsf{P}(T_{\rm e} > K/(\lambda-1)). \tag{4.4}$$

In addition, if (iv) each for $m \in \mathbb{Z}$, $T_m - 1$ *is a stopping time with respect to* $\{N_{m,n}; n = 0,1,\dots\}$*; and (v)* $\boldsymbol{\alpha}(k) = \boldsymbol{\phi}\boldsymbol{\Lambda}(k)$ *for* $k = 1,2,\dots$*, then*

$$P_{\rm loss}^{(K)} \stackrel{K}{\sim} \frac{\lambda-1}{\lambda} \mathsf{P}(T_{\rm e} > K/(\lambda-1)). \tag{4.5}$$

Remark 4.3 Condition (v) implies that BMAPs in on-periods are stationary and thus $\mathsf{P}(J_{m,n} = j) = \phi_j$ ($j \in \mathbb{D}$) for all $m \in \mathbb{Z}$ and $n = 0,1,\dots,T_m - 1$.

Proof of Corollary 4.1. We first show that the cumulative process $\{B(t)\}$ in (4.3) satisfies the conditions of Theorem 3.1. To do so, it suffices to show that condition (ii) of Theorem 3.1 is satisfied. It follows from Assumption 4.1 and Lemma 3.1 that for $n = 0, 1$,

$$\mathsf{P}(\Delta B_n > x) \le C\mathsf{P}(Y > x), \quad \text{for all } x \ge 0. \tag{4.6}$$

Thus condition (iii) of Corollary 4.1 implies $\mathsf{E}[\exp\{Q(\Delta B_n)\}] < \infty$ ($n = 0,1$). Further, $\mathsf{E}[\exp\{Q(\Delta \tau_n)\}] < \infty$ ($n = 0,1$) because the distribution of $\Delta\tau_n$ is phase-type. Therefore, condition (ii) of Theorem 3.1 holds.

Next, applying Theorem 3.1 to the cumulative process $\{B(t)\}$ in (4.3) with sampling time $T_m - 1$ (see (4.2)), we obtain

$$\mathsf{P}(A > x) \stackrel{x}{\sim} \mathsf{P}(T - 1 > x/(\lambda-1)) = \mathsf{P}(T > x/(\lambda-1) + 1),$$

which implies that $\mathsf{E}[A] < \infty$ and

$$\mathsf{P}(A_{\rm e} > x) \stackrel{x}{\sim} \frac{(\lambda-1)\mathsf{E}[T]}{\mathsf{E}[A]} \mathsf{P}(T_{\rm e} > x/(\lambda-1)).$$

Combining this with Proposition 4.1 yields (4.4).

Finally, we prove (4.5). Taking the expectation of both sides of (4.2) and using the dominated convergence theorem, we have

$$\begin{aligned}
\mathsf{E}[A_m + T_m] &= \mathsf{E}\left[\sum_{n=0}^{T_m-1} N_{m,n}\right] = \mathsf{E}\left[\sum_{n=0}^{\infty} \mathbb{1}(T_m - 1 \ge n) N_{m,n}\right] \\
&= \sum_{n=0}^{\infty} \mathsf{E}[\mathbb{1}(T_m - 1 \ge n) N_{m,n}] \\
&= \mathsf{E}[N_{m,0}] + \sum_{n=1}^{\infty} \mathsf{E}[\mathbb{1}(T_m - 1 \ge n) N_{m,n}]. \qquad (4.7)
\end{aligned}$$

It follows from condition (iv) of Corollary 4.1 that $\mathbb{1}(T_m - 1 \ge n) = 1 - \mathbb{1}(T_m - 1 \le n-1)$ is a function of $\{N_{m,0}, N_{m,1}, \dots, N_{m,n-1}\}$ and thus given $J_{m.n-1}$, $N_{m,n}$ is conditionally independent of $\{T_m - 1 \ge n\}$. Therefore, for all $m \in \mathbb{Z}$ and $n = 1, 2, \dots,$

$$\begin{aligned}
&\mathsf{E}[\mathbb{1}(T_m - 1 \ge n) N_{m,n}] \\
&\quad = \sum_{j \in \mathbb{D}} \mathsf{P}(J_{m,n-1} = j) \mathsf{E}[\mathbb{1}(T_m - 1 \ge n) N_{m,n} \mid J_{m,n-1} = j] \\
&\quad = \mathsf{P}(T_m - 1 \ge n) \sum_{j \in \mathbb{D}} \mathsf{P}(J_{m,n-1} = j) \mathsf{E}[N_{m,n} \mid J_{m,n-1} = j] \\
&\quad = \mathsf{P}(T_m - 1 \ge n) \mathsf{E}[N_{m,n}]. \qquad (4.8)
\end{aligned}$$

Condition (v) of Corollary 4.1 and (4.1) yield (see Remark 4.3)

$$\mathsf{E}[N_{m,n}] = \boldsymbol{\phi} \sum_{k=1}^{\infty} k \boldsymbol{\Lambda}(k) \boldsymbol{e} = \lambda, \quad \text{for all } m \in \mathbb{Z},\ n = 0, 1, \dots,$$

from which and (4.8) we have $\mathsf{E}[\mathbb{1}(T_m - 1 \ge n) N_{m,n}] = \lambda \mathsf{P}(T_m - 1 \ge n)$. It thus follows from (4.7) that

$$\mathsf{E}[A_m + T_m] = \lambda \sum_{n=0}^{\infty} \mathsf{P}(T_m - 1 \ge n) = \lambda \mathsf{E}[T]. \qquad (4.9)$$

Substituting (4.9) into (4.4) yields (4.5). □

Remark 4.4 Condition (iv) of Corollary 4.1 is used to obtain (4.8). However, according to Wald's lemma (see, e.g., [7, Chapter 1, Theorem 3.2]), this condition is weakened to that $\{T_m \ge n+1\}$ is independent of $N_{m,n}$ for $m \in \mathbb{Z}$ and $n = 0, 1, \dots$.

Corollary 4.2 *Suppose Assumption 4.1 holds and* $\mathsf{E}[A] < \mathsf{E}[I^{\rm off}]$. *Further, suppose (i)* $T \in \mathcal{C}$*; (ii)* $\mathsf{E}[T] < \infty$*; and (iii)* $x\mathsf{P}(Y > x) = o(\mathsf{P}(T > x))$. *We then have (4.4). In addition, if each for* $m \in \mathbb{Z}$, $T_m - 1$ *is a stopping time with respect to* $\{N_{m,n}; n = 0, 1, \dots\}$ *and* $\boldsymbol{\alpha}(k) = \boldsymbol{\phi}\boldsymbol{\Lambda}(k)$ *for* $k = 1, 2, \dots$, *then (4.5) holds.*

Proof. Since $\mathcal{C} \subset \mathcal{L} \cap \mathcal{D}$ (see Remark 2.4), conditions (i) and (ii) imply $T_{\rm e} \in \mathcal{S}$ (see Theorem 3.2 in [27]). Thus, if the conditions of Theorem 3.2 hold, we can readily obtain (4.4) and thus (4.5) by following the proof of Corollary 4.1 (and using Theorem 3.2 instead of Theorem 3.1). Therefore, in what follows, we confirm that the conditions of Theorem 3.2 are satisfied.

Since the distribution of $\Delta\tau_n$ ($n = 0, 1$) is phase-type,

$$\mathsf{E}[(\Delta\tau_n)^p] < \infty, \quad \text{for all } p > 0.$$

Thus, since $T \in \mathcal{C}$, Proposition 2.1 implies that for some $\gamma > 0$,

$$\limsup_{x\to\infty} \frac{x\mathsf{P}(\Delta\tau_n > x)}{\mathsf{P}(T > x)} \le C \limsup_{x\to\infty} x^{\gamma+1}\mathsf{P}(\Delta\tau_n > x) = 0.$$

Further, it follows from Assumption 4.1, Lemma 3.1 and condition (iii) of Corollary 4.2 that

$$x\mathsf{P}(\Delta B_n > x) = O(x\mathsf{P}(Y > x)) = o(\mathsf{P}(T > x)).$$

With these results, we obtain

$$\begin{aligned} x\mathsf{P}(|\Delta B_1 - \Delta\tau_1| > x) &\le x[\mathsf{P}(\Delta B_1 > x) + \mathsf{P}(\Delta\tau_1 > x)] \\ &= o(\mathsf{P}(T > x)). \end{aligned}$$

As a result, it is shown that conditions (i)–(iv) of Theorem 3.2 are satisfied. As for condition (v) of Theorem 3.2, condition (v.b) is satisfied by $\mathsf{E}[T] < \infty$. □

Corollary 4.3 *Suppose Assumption 4.1 holds,* $\mathsf{E}[A] < \mathsf{E}[I^{\rm off}]$ *and* T_m *is independent of the* m*th BMAP for all* $m \in \mathbb{Z}$*. Further, suppose (i)* $T \in \mathcal{L}^{1/\theta}$ *for some* $0 < \theta \le 1/2$*; (ii)* $\mathsf{E}[T] < \infty$ *and* $T_{\rm e} \in \mathcal{S}$*; and (iii)* $\mathsf{E}[\exp\{Q(Y)\}] < \infty$ *for some* $Q \in \mathcal{SC}$ *such that* $x^\theta = O(Q(x))$*. We then have (4.4). In addition, if* $\boldsymbol{\alpha}(k) = \boldsymbol{\phi}\boldsymbol{\Lambda}(k)$ *for* $k = 1, 2, \dots$*, then (4.5) holds.*

Proof. Note first that if T_m ($m \in \mathbb{Z}$) is independent of the mth BMAP, then T_m is a stopping time with respect to $\{N_{m,n}; n = 0, 1, \dots\}$. Thus, the proofs of Corollaries 4.1 and 4.2 imply that to prove (4.4) and (4.5), it suffices to show that the conditions of Theorem 3.3 are satisfied. We already know that $\mathsf{E}[(\Delta\tau_1)^2] < \infty$. Further, since the cumulative process $\{B(t)\}$ in (4.3) is nondecreasing with t, we have $\Delta B_n^* = \Delta B_n \ge 0$ ($n = 0, 1$). It thus follows from (4.6) and condition (iii) of Corollary 4.3 that $\mathsf{E}[\exp\{Q(\Delta B_n^*)\}] < \infty$ ($n = 0, 1$), which leads to $\mathsf{E}[(\Delta B_1)^2] < \infty$. As a result, the conditions of Theorem 3.3 hold. □

The following corollary is proved by using Theorem 3.5.

Corollary 4.4 *Suppose Assumption 4.1 holds,* $\mathsf{E}[A] < \mathsf{E}[I^{\rm off}]$ *and* T_m *is independent of the* m*th BMAP for all* $m \in \mathbb{Z}$*. Further, suppose (i)* $T \in \mathcal{C}$*; (ii)* $\mathsf{E}[T] < \infty$*; and (iii)* $\mathsf{P}(Y > x) = o(\mathsf{P}(T > x))$*. We then have (4.4). In addition, if* $\boldsymbol{\alpha}(k) = \boldsymbol{\phi}\boldsymbol{\Lambda}(k)$ *for* $k = 1, 2, \dots$*, then (4.5) holds.*

Proof. It is easy to see that the conditions of Theorem 3.5 are satisfied. Thus, similarly to the other theorems in this subsection, we can prove (4.4) and (4.5). □

Finally, we mention previous studies related to the results presented in this subsection. Zwart [46] and Jelenković and Momčilović [24] study the subexponential asymptotics of the loss probability of finite-buffer fluid queues fed by the superposition of independent on/off sources that generate fluid at constant rates. These studies assume that active periods of each on/off source are regularly varying or consistently varying, and then derive asymptotic formulas for the loss probability such that the decay of the loss probability is connected to the tail of the equilibrium distribution of active periods.

A Preliminary Lemmas

This appendix presents preliminary lemmas, whose proofs are all given in Appendix B.

A.1 Higher-order long-tailed distributions

Lemma A.1 *If* $X \in \mathcal{L}^{1/\theta}$ *(i.e.,* $X^\theta \in \mathcal{L}$*) for some* $0 < \theta \le 1$*, the following are satisfied:*

(i) $\lim_{x\to\infty} e^{\varepsilon x^\theta} \mathsf{P}(X > x) = \infty$ *for any* $\varepsilon > 0$*, i.e.,* $\mathsf{P}(X > x) = e^{-o(x^\theta)}$*.*

(ii) $X \in \mathcal{L}^{1/\eta}$ *for all* $1 \le 1/\eta < 1/\theta$*.*

Remark A.1 Lemma A.1 (ii) implies that $\mathcal{L}^{p_2} \subset \mathcal{L}^{p_1}$ for $1 \le p_1 < p_2$.

Lemma A.2 *For any* $0 < \theta \le 1$*,* $X \in \mathcal{L}^{1/\theta}$ *if and only if* $\mathsf{P}(X > x - \xi x^{1-\theta}) \overset{x}{\sim} \mathsf{P}(X > x)$ *for all* $\xi \in \mathbb{R}$*.*

Lemma A.2 is an extension of Lemma 1 in [25]. The following lemma shows that the if-part of Lemma A.2 holds under a weaker condition.

Lemma A.3 *For any* $0 < \theta \le 1$*,* $X \in \mathcal{L}^{1/\theta}$ *if* $\mathsf{P}(X > x - \xi x^{1-\theta}) \overset{x}{\sim} \mathsf{P}(X > x)$ *for some* $\xi \in \mathbb{R}\backslash\{0\}$.

Remark A.2 The statements of Lemmas A.1–A.3 are presented in a slightly different way in a technical report [34] (see Lemmas 1–3 therein), where the statements are described in terms of h-sensitivity (see Chapter 2 in [16] for the definition of h-sensitivity).

Lemma A.4 *Suppose* $X \in \mathcal{L}^{1/\theta}$ *for some* $0 < \theta \le 1$*. Let* g *denote a nonnegative function on* $[0,\infty)$ *such that* $\limsup_{x\to\infty} g(x)/x < 1$*. Then for any* $\varepsilon > 0$*, there exists* $\breve{x}_\varepsilon > 0$ *such that for all* $x > \breve{x}_\varepsilon$ *and* $0 \le u \le g(x)$*,*

$$\mathsf{P}(X > x - u) \le \mathsf{P}(X > x) e^{\varepsilon(u^\theta + 1)}.$$

Lemma A.5 *$\mathcal{C} \subset \mathcal{L}^{\infty}$, i.e., $\mathcal{C} \subset \mathcal{L}^{1/\theta}$ for any $0 < \theta \le 1$.*

A.2 Subexponential concave distributions

The subexponential concave class was first introduced by Nagaev [35]. According to Nagaev's definition of $\mathcal{SC}$, condition (iii) of Definition 2.2 is replaced by the following condition: $(\mathrm{iii})'$ there exist $x_0 > 0$, $0 < \alpha < 1$ and $0 < \beta < 1$ such that for all $x \ge x_0$ and $\beta x \le u \le x$,

$$\frac{Q_X(x) - Q_X(u)}{Q_X(x)} \le \alpha \frac{x-u}{x}. \tag{A.1}$$

Actually, Nagaev's definition is equivalent to Definition 2.2. Lemma 3.1 (i) in [23] shows that Nagaev's definition implies Definition 2.2. The converse follows from Theorem 2 in [39], though the phrase "$Q(x)/f(x)$ is nondecreasing" should be replaced by "$Q(x)/f(x)$ is nonincreasing".

Remark A.3 Suppose $Q \in \mathcal{SC}$ is differentiable. It then follows from (A.1) and (2.1) that

$$Q'(x) := \frac{\mathrm{d}}{\mathrm{d}x} Q(x) \le \frac{\alpha Q(x)}{x} \le C x^{\alpha - 1}. \tag{A.2}$$

It is known that $\mathcal{SC} \subset \mathcal{S}^*$ (see Lemma 1 in [38]), where $\mathcal{S}^*$ is a subclass of $\mathcal{S}$ and is defined as follows (see, e.g., [20]): A d.f. F on $[0, \infty)$ is said to be in $\mathcal{S}^*$ if F has a finite positive mean μ and

$$\lim_{x\to\infty} \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)} \overline{F}(y) \mathrm{d}y = 2\mu.$$

An important property of $\mathcal{S}^*$ is that $F \in \mathcal{S}^*$ implies $F, F_{\mathrm{e}} \in \mathcal{S}$, where F_{e} denotes the equilibrium distribution (or integrated tail distribution) of F, i.e., $F_{\mathrm{e}}(x) = \mu^{-1} \int_0^x \overline{F}(y)\mathrm{d}y$ for $x \ge 0$.

The following lemma establishes the relationship between class $\mathcal{SC}$ and the higher-order long-tailed class.

Lemma A.6 *(i) $\mathcal{SC}_{\alpha} \subset \mathcal{L}^{1/\beta}$ for all $0 < \alpha < \beta \le 1$.*

(ii) $X^{\alpha} \in \mathcal{L}$ if $X \in \mathcal{SC}_{\alpha}$ for some $0 < \alpha < 1$ and

$$\lim_{x\to\infty} Q_X(x)/x^{\alpha} = 0. \tag{A.3}$$

A.3 Regular varying distributions

The definition of class $\mathcal{R}$ is as follows:

Definition A.1 A nonnegative r.v. X and its d.f. F_X are said to be regularly varying with index $-\alpha$ ($\alpha > 0$) if

$$\lim_{x\to\infty} \frac{\overline{F}_X(vx)}{\overline{F}_X(x)} = v^{-\alpha}, \quad \text{for all } v > 0.$$

Further, the class of such distributions is denoted by $\mathcal{R}_{-\alpha}$, and the regular variation class $\mathcal{R}$ is given by $\cup_{\alpha>0}\mathcal{R}_{-\alpha}$.

Remark A.4 Every regular varying distribution F can be expressed as $\overline{F}(x) = x^{-\alpha}L(x)$ for some $\alpha > 0$, where L is a slowly varying function, i.e., $L \in \mathcal{R}_0$ (see section 1.5 in [6]).

Remark A.5 It is known that $\mathcal{R} \subset \mathcal{C}$ (see, e.g., [11, 12]). Thus $\mathcal{R} \subset \mathcal{C} \subset \mathcal{L}^\infty \subset \mathcal{L}^p \subset \mathcal{L}$ for any $p > 1$ (see Remark A.1 and Lemma A.5).

The following lemma is used to prove Theorem 3.2.

Lemma A.7 *Let U denote an r.v. in $\mathbb{R}$ with $\mathsf{E}[|U|] < \infty$. Suppose $\mathsf{P}(U > x) = o(\mathsf{P}(Y > x))$ for some nonnegative r.v. Y with $\mathsf{E}[Y] < \infty$. Then for any $\mu > \mathsf{E}[U]$, there exists some r.v. Z in $\mathbb{R}$ such that $\mathsf{E}[Z] = \mu$, $Z \ge U$ w.p.1, and for all sufficiently large $x > 0$,*

$$\overline{F}_Z(x) = l_0(x)\overline{F}_Y(x),$$

where l_0 is some slowly varying function such that $\lim_{x\to\infty} l_0(x) = 0$.

A.4 Bounds on the deviation probabilities

Lemma A.8 *Let X_n's ($n = 1, 2, \dots$) denote independent copies of a nonnegative r.v. X with $\mathsf{E}[X] > 0$ and $\mathsf{E}[X^2] < \infty$. Let $N_X(x) = \max\{k \ge 0; \sum_{n=1}^k X_n \le x\}$ for $x \ge 0$. Then for any $\delta > 0$, there exist some constants $\widetilde{C} := \widetilde{C}(\delta) > 0$ and $\widetilde{c} := \widetilde{c}(\delta) > 0$ such that*

$$\mathsf{P}\left(N_X(x) - \frac{x}{\mathsf{E}[X]} > u\right) \le \widetilde{C}\exp\{-\widetilde{c}u^2/x\}, \quad \forall x \ge 0,\ \ 0 \le \forall u \le \delta x. \tag{A.4}$$

Remark A.6 Lemma A.8 is very similar to, but not exactly the same as Lemma 6 in [25]. The latter states that there exists *some* $\delta > 0$ such that (A.4) holds.

The following lemmas are extensions of Lemma 2.3 in [43] and Lemma 2.2 in [30] to the maxima of partial sums of i.i.d. r.v.s.

Lemma A.9 *Let U_n's ($n = 1, 2, \dots$) denote independent copies of an r.v. U in $\mathbb{R}$, which satisfies $\mathsf{E}[U] = 0$ and $\mathsf{E}[(U^+)^r] < \infty$ for some $r > 1$. Then for any fixed $\gamma > 0$ and $p > 0$, there exist some $v := v(r,p) > 0$ and $\widetilde{C} := \widetilde{C}(v,\gamma)$ such that for all $n = 1, 2, \dots,$*

$$\mathsf{P}\left(\max_{1\le k\le n} \sum_{i=1}^k U_i \ge x\right) \le n\mathsf{P}(U > vx) + \widetilde{C}x^{-p}, \quad \forall x \ge \gamma n. \tag{A.5}$$

Lemma A.10 *Let U_n's ($n = 1, 2, \dots$) denote independent copies of an r.v. U in $\mathbb{R}$. Suppose $0 \le \mathsf{E}[U] < \infty$ and $U^+ \in \mathcal{C}$. Then for any $\gamma > \mathsf{E}[U]$, there exists some constant $\widetilde{C} := \widetilde{C}(\gamma) > 0$ such that for all $n = 1, 2, \dots$,*

$$\mathsf{P}\left(\max_{1 \le k \le n} \sum_{i=1}^{k} U_i > x \right) \le \widetilde{C} n \mathsf{P}(U > x), \quad \forall x \ge \gamma n. \tag{A.6}$$

A.5 Convolution tail of matrix-valued functions associated with subexponential distributions

Let $\boldsymbol{F} = (F_{i,j})$ and $\boldsymbol{G} = (G_{i,j})$ denote $m_0 \times m_1$ and $m_1 \times m_2$ matrix-valued functions on $\mathbb{R}$. Assume that $F_{i,j}(x)$ and $G_{i,j}(x)$ are nonnegative and nondecreasing for all $x \in \mathbb{R}$ and that $F_{i,j}(\infty) := \lim_{x\to\infty} F_{i,j}(x) < \infty$ and $G_{i,j}(\infty) := \lim_{x\to\infty} G_{i,j}(x) < \infty$. We then define $\overline{\boldsymbol{F}}(x) = \boldsymbol{F}(\infty) - \boldsymbol{F}(x)$ and $\overline{\boldsymbol{G}}(x) = \boldsymbol{G}(\infty) - \boldsymbol{G}(x)$ for $x \in \mathbb{R}$.

Let $\boldsymbol{F} * \boldsymbol{G}$ denote the convolution of $\boldsymbol{F}$ and $\boldsymbol{G}$, i.e.,

$$\boldsymbol{F} * \boldsymbol{G}(x) = \int_{y\in\mathbb{R}} \boldsymbol{F}(x-y)\mathrm{d}\boldsymbol{G}(y) = \int_{y\in\mathbb{R}} \mathrm{d}\boldsymbol{F}(y)\boldsymbol{G}(x-y), \qquad x \in \mathbb{R}.$$

Let $\overline{\boldsymbol{F} * \boldsymbol{G}}(x)$ ($x \in \mathbb{R}$) denote

$$\overline{\boldsymbol{F} * \boldsymbol{G}}(x) = \boldsymbol{F} * \boldsymbol{G}(\infty) - \boldsymbol{F} * \boldsymbol{G}(x) = \boldsymbol{F}(\infty)\boldsymbol{G}(\infty) - \boldsymbol{F} * \boldsymbol{G}(x).$$

When $\boldsymbol{F}$ is a square matrix-valued function (i.e., $m_0 = m_1$), we define $\boldsymbol{F}^{*n}$ ($n = 1, 2, \dots$) as the n-fold convolution of $\boldsymbol{F}$ itself, i.e.,

$$\boldsymbol{F}^{*n}(x) = \boldsymbol{F}^{*(n-1)} * \boldsymbol{F}(x), \quad x \in \mathbb{R},$$

and for convenience, define $\boldsymbol{F}^{*0}(x)$ as zero matrix (denoted by $\boldsymbol{O}$) for $x < 0$ and $\boldsymbol{F}^{*0}(x) = \boldsymbol{I}$ for $x \ge 0$. Further, for the n-fold convolution $\boldsymbol{F}^{*n}$, let $\overline{\boldsymbol{F}^{*n}}(x)$ ($x \in \mathbb{R}$) denote

$$\overline{\boldsymbol{F}^{*n}}(x) = \boldsymbol{F}^{*n}(\infty) - \boldsymbol{F}^{*n}(x) = (\boldsymbol{F}(\infty))^n - \boldsymbol{F}^{*n}(x).$$

We then have the following lemma, which is the upper-limit version of Proposition A.3 in [33] and Lemma 6 in [21].

Lemma A.11 *Suppose for some r.v. $Y \in \mathcal{S}$,*

$$\limsup_{x\to\infty} \frac{\overline{\boldsymbol{F}}(x)}{\mathsf{P}(Y > x)} \le \widetilde{\boldsymbol{F}}, \qquad \limsup_{x\to\infty} \frac{\overline{\boldsymbol{G}}(x)}{\mathsf{P}(Y > x)} \le \widetilde{\boldsymbol{G}}, \tag{A.7}$$

where $\widetilde{\boldsymbol{F}} = (\widetilde{F}_{i,j})$ and $\widetilde{\boldsymbol{G}} = (\widetilde{G}_{i,j})$ are finite, and where $\widetilde{\boldsymbol{F}} = \widetilde{\boldsymbol{G}} = \boldsymbol{O}$ is allowed. We then have

$$\limsup_{x\to\infty} \frac{\overline{\boldsymbol{F} * \boldsymbol{G}}(x)}{\mathsf{P}(Y > x)} \le \widetilde{\boldsymbol{F}}\boldsymbol{G}(\infty) + \boldsymbol{F}(\infty)\widetilde{\boldsymbol{G}}. \tag{A.8}$$

Further, suppose $\boldsymbol{F}$ is a square matrix-valued function. Then

$$\limsup_{x\to\infty} \frac{\overline{\boldsymbol{F}^{*n}}(x)}{\mathsf{P}(Y > x)} \le \sum_{l=0}^{n-1} (\boldsymbol{F}(\infty))^l \widetilde{\boldsymbol{F}} (\boldsymbol{F}(\infty))^{n-l-1}. \tag{A.9}$$

In addition, if $\sum_{n=0}^{\infty} (\boldsymbol{F}(\infty))^n$ is finite, then

$$\limsup_{x\to\infty} \sum_{n=0}^{\infty} \frac{\overline{\boldsymbol{F}^{*n}}(x)}{\mathsf{P}(Y > x)} \le (\boldsymbol{I} - \boldsymbol{F}(\infty))^{-1} \widetilde{\boldsymbol{F}} (\boldsymbol{I} - \boldsymbol{F}(\infty))^{-1}. \tag{A.10}$$

B Proofs of Lemmas

B.1 Proof of Lemma 2.1

To prove Lemma 2.1, we consider another (possibly delayed) cumulative process $\{\breve{B}(t); t \ge 0\}$ on $\mathbb{R}$, which satisfies $|\breve{B}(0)| < \infty$ w.p.1 and has the same regenerative points as $\{B(t); t \ge 0\}$, i.e., τ_n's ($n = 0, 1, \dots$). Let $\Delta\breve{B}_n = \breve{B}(\tau_n) - \breve{B}(\tau_{n-1})$ and $\Delta\breve{B}_n^* = \sup_{\tau_{n-1} \le t \le \tau_n} \breve{B}(t) - \breve{B}(\tau_{n-1})$ for $n = 0, 1, \dots$. The $\Delta\breve{B}_n$'s (resp. $\Delta\breve{B}_n^*$'s) ($n = 1, 2, \dots$) are i.i.d. and independent of $\Delta\breve{B}_0$ (resp. $\Delta\breve{B}_0^*$). We assume that

$$\mathsf{P}(0 \le \breve{B}_n^* < \infty) = 1 \ \ (n = 0, 1), \quad \mathsf{E}[|\Delta\breve{B}_1|] < \infty.$$

Unlike $\{B(t)\}$, the expected increment of $\{\breve{B}(t)\}$ per regenerative cycle, i.e., $\mathsf{E}[\Delta\breve{B}_1]$, is not necessarily assumed to be positive.

The following lemma is an extension of Proposition 1 in [25]. Using the lemma, we can readily prove Lemma 2.1.

Lemma B.1 *Suppose $\mathsf{E}[(\Delta\breve{B}_1)^2] < \infty$ and $\breve{b} := \mathsf{E}[\Delta\breve{B}_1]/\mathsf{E}[\Delta\tau_1]$ is finite and nonzero. Let $\Delta\Omega_n = \Delta\breve{B}_n^* - \min(\breve{b}, 0)\Delta\tau_n \ge 0$ for $n = 0, 1, \dots$. If $\mathsf{E}[(\Delta\tau_1)^2] < \infty$ and $\mathsf{E}[\exp\{Q(\Delta\Omega_n)\}] < \infty$ ($n = 0, 1$) for some $Q \in \mathcal{SC}$, then for all $x, u \ge 0$,*

$$\mathsf{P}\left(\sup_{0 \le t \le x} \{\breve{B}(t) - \breve{b}t\} > u \right) \le C \left(e^{-cu^2/x} + e^{-cx} + x e^{-cQ(u)} \right), \tag{B.1}$$

where C and c are independent of x and u.

Note that if $\breve{B}(t) = B(t)$, then $\breve{b} = b > 0$, $\Delta\Omega_n = \Delta B_n^*$ and

$$\mathsf{P}\left(\sup_{0 \le t \le x} \{\breve{B}(t) - \breve{b}t\} > u \right) = \mathsf{P}\left(\sup_{0 \le t \le x} \{B(t) - bt\} > u \right).$$

On the other hand, suppose $\breve{B}(t) = -B(t)$. We then $\breve{b} = -b < 0$ and

$$\begin{aligned} \Delta\Omega_n &= \sup_{\tau_{n-1} \le t \le \tau_n} (B(\tau_{n-1}) - B(t)) + b\Delta\tau_n \\ &=: \Delta B_n^{**} + b\Delta\tau_n \ge b\Delta\tau_n. \end{aligned}$$

Thus $\mathsf{E}[\exp\{Q(\Delta\Omega_1)\}] < \infty$ implies $\mathsf{E}[(\Delta\tau_1)^2] < \infty$ (see Remark 3.2). We also have

$$\mathsf{P}\left(\sup_{0\le t\le x}\{\breve{B}(t) - \breve{b}t\} > u\right) = \mathsf{P}\left(\inf_{0\le t\le x}\{B(t) - bt\} < -u\right).$$

As a result, Lemma 2.1 follows from Lemma B.1. As for the proof of Lemma B.1, see Appendix B.16.

B.2 Proof of Lemma 3.1

We first partition $\widetilde{\boldsymbol{\beta}}$ and $\widetilde{\boldsymbol{H}}$ as

$$\widetilde{\boldsymbol{\beta}} = \begin{array}{c} \begin{array}{cc} \{0\} & \mathbb{D}\setminus\{0\} \end{array} \\ \begin{pmatrix} \widetilde{\beta}_0 & \widetilde{\boldsymbol{\beta}}_+ \end{pmatrix} \end{array}, \quad \widetilde{\boldsymbol{H}} = \begin{array}{c c} & \begin{array}{cc} \{0\} & \mathbb{D}\setminus\{0\} \end{array} \\ \begin{array}{c} \{0\} \\ \mathbb{D}\setminus\{0\} \end{array} & \begin{pmatrix} \widetilde{H}_{0,0} & \widetilde{\boldsymbol{\eta}}_+ \\ \widetilde{\boldsymbol{h}}_+ & \widetilde{\boldsymbol{H}}_+ \end{pmatrix} \end{array}.$$

Then, letting $z = 1$ in (3.3) and (3.4) and taking the inverse of them with respect to ξ, we have

$$\mathsf{P}(\Delta B_0 \le x) = \beta_0(x) + \boldsymbol{\beta}_+ * \sum_{n=0}^{\infty} \boldsymbol{H}_+^{*n} * \boldsymbol{h}_+(x), \tag{B.2}$$

$$\mathsf{P}(\Delta B_1 \le x) = H_{0,0}(x) + \boldsymbol{\eta}_+ * \sum_{n=0}^{\infty} \boldsymbol{H}_+^{*n} * \boldsymbol{h}_+(x), \tag{B.3}$$

where "$*$" denotes the operator of convolution and the superscript "$*n$" represents the nth-fold convolution (see Appendix A.5). Applying Lemma A.11 to (B.2) and (B.3), we obtain

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\mathsf{P}(\Delta B_0 > x)}{\mathsf{P}(Y > x)} &\le c^*\widetilde{\beta}_0 + c^*\widetilde{\boldsymbol{\beta}}_+(\boldsymbol{I} - \widehat{\boldsymbol{H}}_+)^{-1}\widehat{\boldsymbol{h}}_+ \\
&\quad + \widehat{\boldsymbol{\beta}}_+(\boldsymbol{I} - \widehat{\boldsymbol{H}}_+)^{-1}(c^*\widetilde{\boldsymbol{H}}_+)(\boldsymbol{I} - \widehat{\boldsymbol{H}}_+)^{-1}\widehat{\boldsymbol{h}}_+ \\
&\quad + \widehat{\boldsymbol{\beta}}_+(\boldsymbol{I} - \widehat{\boldsymbol{H}}_+)^{-1}(c^*\widetilde{\boldsymbol{h}}_+) \\
&= c^*\left[\widetilde{\boldsymbol{\beta}}\boldsymbol{e} + \widehat{\boldsymbol{\beta}}_+(\boldsymbol{I} - \widehat{\boldsymbol{H}}_+)^{-1}(\widetilde{\boldsymbol{H}}_+\boldsymbol{e} + \widetilde{\boldsymbol{h}}_+)\right] \le c^*C,
\end{aligned}$$

and

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\mathsf{P}(\Delta B_1 > x)}{\mathsf{P}(Y > x)} &\le c^*\widetilde{H}_{0,0} + c^*\widetilde{\boldsymbol{\eta}}_+(\boldsymbol{I} - \widehat{\boldsymbol{H}}_+)^{-1}\widehat{\boldsymbol{h}}_+ \\
&\quad + \widehat{\boldsymbol{\eta}}_+(\boldsymbol{I} - \widehat{\boldsymbol{H}}_+)^{-1}(c^*\widetilde{\boldsymbol{H}}_+)(\boldsymbol{I} - \widehat{\boldsymbol{H}}_+)^{-1}\widehat{\boldsymbol{h}}_+ \\
&\quad + \widehat{\boldsymbol{\eta}}_+(\boldsymbol{I} - \widehat{\boldsymbol{H}}_+)^{-1}(c^*\widetilde{\boldsymbol{h}}_+) \\
&= c^*\left[\widetilde{H}_{0,0} + \widetilde{\boldsymbol{\eta}}_+\boldsymbol{e} + \widehat{\boldsymbol{\eta}}_+(\boldsymbol{I} - \widehat{\boldsymbol{H}}_+)^{-1}(\widetilde{\boldsymbol{H}}_+\boldsymbol{e} + \widetilde{\boldsymbol{h}}_+)\right] \\
&= c^*(1/\varpi_0)\left[\varpi_0(\widetilde{H}_{0,0} + \widetilde{\boldsymbol{\eta}}_+\boldsymbol{e}) + \boldsymbol{\varpi}_+(\widetilde{\boldsymbol{H}}_+\boldsymbol{e} + \widetilde{\boldsymbol{h}}_+)\right] \\
&= c^*(1/\varpi_0)\boldsymbol{\varpi}\widetilde{\boldsymbol{H}}\boldsymbol{e} \le c^*C,
\end{aligned}$$

where we use $(\boldsymbol{I} - \widehat{\boldsymbol{H}}_+)^{-1}\widehat{\boldsymbol{h}}_+ = \boldsymbol{e}$ (which is due to $\widehat{\boldsymbol{h}}_+ + \widehat{\boldsymbol{H}}_+\boldsymbol{e} = \boldsymbol{e}$); and also use $\boldsymbol{\varpi}_+ := (\varpi_i)_{i\in\mathbb{D}\setminus\{0\}} = \varpi_0\widehat{\boldsymbol{\eta}}_+(\boldsymbol{I} - \widehat{\boldsymbol{H}}_+)^{-1}$ and $\varpi_0 = 1/\mathsf{E}[\Delta\tau_1]$ (see, e.g., [7, Chapter 3, Theorems 2.1 and 3.2]).

B.3 Proof of Lemma 3.2

Let $\psi_1(k,\xi)$ ($k = 0,1,\dots$) denote

$$\psi_1(k,\xi) = \mathsf{E}[\mathbb{1}(\Delta\tau_1 = k)e^{\mathrm{i}\xi\Delta B_1}].$$

It then follows from (3.4) that for $k = 1,2,\dots$.

$$\begin{aligned}
\psi_1(k,\xi) &= \frac{1}{k!}\,\frac{\partial^k}{\partial z^k}\widehat{\psi}_1(z,\xi)\bigg|_{z=0} \\
&= \mathbb{1}(k=1)\widehat{H}_{0,0}(\xi) \\
&\quad + \mathbb{1}(k\ge 2)\frac{1}{k!}\widehat{\boldsymbol{\eta}}_+(\xi)\cdot\frac{k!}{(k-2)!}\frac{\partial^{k-2}}{\partial z^{k-2}}\left(\boldsymbol{I} - z\widehat{\boldsymbol{H}}_+(\xi)\right)^{-1}\bigg|_{z=0}\cdot\widehat{\boldsymbol{h}}_+(\xi) \\
&= \mathbb{1}(k=1)\widehat{H}_{0,0}(\xi) + \mathbb{1}(k\ge 2)\widehat{\boldsymbol{\eta}}_+(\xi)\cdot(\widehat{\boldsymbol{H}}_+(\xi))^{k-2}\cdot\widehat{\boldsymbol{h}}_+(\xi).
\end{aligned} \tag{B.4}$$

Taking the inverse of (B.4) with respect to ξ and applying Lemma A.11 to the resulting equation, we obtain

$$\begin{aligned}
&\limsup_{x\to\infty}\frac{\mathsf{P}(\Delta\tau_1 = k, \Delta B_1 > x)}{\mathsf{P}(Y > x)} \\
&\quad\le c^*\mathbb{1}(k=1)\widetilde{H}_{0,0} + c^*\mathbb{1}(k\ge 2)\left\{\widetilde{\boldsymbol{\eta}}_+(\widehat{\boldsymbol{H}}_+)^{k-2}\widehat{\boldsymbol{h}}_+\right\} \\
&\qquad + c^*\mathbb{1}(k\ge 2)\left\{\widehat{\boldsymbol{\eta}}_+\cdot\sum_{l=0}^{k-3}(\widehat{\boldsymbol{H}}_+)^l\widetilde{\boldsymbol{H}}_+(\widehat{\boldsymbol{H}}_+)^{k-l-3}\cdot\widehat{\boldsymbol{h}}_+\right\} \\
&\qquad + c^*\mathbb{1}(k\ge 2)\left\{\widehat{\boldsymbol{\eta}}_+(\widehat{\boldsymbol{H}}_+)^{k-2}\widetilde{\boldsymbol{h}}_+\right\}, \quad \text{for all } k = 1,2,\dots.
\end{aligned} \tag{B.5}$$

Note here that $\widehat{\beta}_i = 0$ (resp. $\widehat{H}_{i,j} = 0$) implies $\widetilde{\beta}_i = 0$ (resp. $\widetilde{H}_{i,j} = 0$). Thus we can fix $\widetilde{\boldsymbol{\beta}} \le C\widehat{\boldsymbol{\beta}}$ and $\widetilde{\boldsymbol{H}} \le C\widehat{\boldsymbol{H}}$. Therefore, from (3.5) and (B.5), we have for all $k = 1,2,\dots$,

$$\begin{aligned}
&\mathsf{P}(\Delta\tau_1 = k, \Delta B_1 > x) \\
&\quad\lesssim_x c^*C\left[\mathbb{1}(k=1)k\widehat{H}_{0,0} + \mathbb{1}(k\ge 2)k\left\{\widehat{\boldsymbol{\eta}}_+(\widehat{\boldsymbol{H}}_+)^{k-2}\widehat{\boldsymbol{h}}_+\right\}\right]\mathsf{P}(Y > x) \\
&\quad= c^*Ck\mathsf{P}(\Delta\tau_1 = k)\mathsf{P}(Y > x),
\end{aligned}$$

where C is independent of k.

B.4 Proof of Lemma 3.3

Under the assumptions made in subsection 3.2, $N(t) = N(\lfloor t \rfloor)$ for all $t \ge 0$. Thus, without loss of generality, we can fix $t = n$, where n is a nonnegative integer.

Since event $\{N(n) = m\}$ is equivalent to $\{\sum_{i=1}^{m} \Delta\tau_i \le n, \sum_{i=1}^{m+1} \Delta\tau_i > n\}$, we have

$$\begin{aligned}
&\mathsf{P}\left(N(n) = m, \sum_{i=1}^{m} \Delta B_i > x\right) \\
&\quad = \mathsf{P}\left(\sum_{i=1}^{m} \Delta\tau_i \le n, \sum_{i=1}^{m+1} \Delta\tau_i > n, \sum_{i=1}^{m} \Delta B_i > x\right) \\
&\quad = \sum_{k=1}^{n} \mathsf{P}\left(\sum_{i=1}^{m} \Delta\tau_i = k, \sum_{i=1}^{m} \Delta B_i > x\right) \mathsf{P}(\Delta\tau_{m+1} > n - k).
\end{aligned} \tag{B.6}$$

Note here that

$$\begin{aligned}
&\mathsf{P}\left(\sum_{i=1}^{m} \Delta\tau_i = k, \sum_{i=1}^{m} \Delta B_i > x\right) \\
&\qquad = \sum_{k_1+\cdots+k_m=k} \prod_{i=1}^{m} \mathsf{P}(\Delta\tau_i = k_i) \\
&\qquad\qquad \times \mathsf{P}\left(\sum_{i=1}^{m} \Delta B_i > x \,\middle|\, \Delta\tau_i = k_i,\ i = 1, 2, \dots, m\right).
\end{aligned} \tag{B.7}$$

Note also that the ΔB_i's ($i = 1, 2, \dots, m$) are conditionally independent given that $\Delta\tau_i = k_i$ ($i = 1, 2, \dots, m$). It thus follows from Lemmas 3.2 and A.11 that for $(k_1, \dots, k_m)$ such that $\sum_{i=1}^{m} k_i = k$,

$$\begin{aligned}
&\mathsf{P}\left(\sum_{i=1}^{m} \Delta B_i > x \,\middle|\, \Delta\tau_i = k_i,\ i = 1, 2, \dots, m\right) \\
&\qquad \lesssim_x (k_1 + \cdots + k_m) \cdot c^* C \mathsf{P}(Y > x) = k \cdot c^* C \mathsf{P}(Y > x).
\end{aligned}$$

Combining this with (B.7) yields

$$\begin{aligned}
&\mathsf{P}\left(\sum_{i=1}^{m} \Delta\tau_i = k, \sum_{i=1}^{m} \Delta B_i > x\right) \\
&\qquad \lesssim_x c^* C k \sum_{k_1+\cdots+k_m=k} \prod_{i=1}^{m} \mathsf{P}(\Delta\tau_i = k_i) \mathsf{P}(Y > x) \\
&\qquad = c^* C k \cdot \mathsf{P}\left(\sum_{i=1}^{m} \Delta\tau_i = k\right) \mathsf{P}(Y > x).
\end{aligned} \tag{B.8}$$

From (B.6) and (B.8), we obtain for all $n = 0, 1, \dots$ and $m = 0, 1, \dots, n$.

$$\begin{aligned}
&\mathsf{P}\left(N(n) = m, \sum_{i=1}^{m} \Delta B_i > x\right) \\
&\lesssim_x c^* C \sum_{k=1}^{n} k \mathsf{P}\left(\sum_{i=1}^{m} \Delta\tau_i = k\right) \mathsf{P}(\Delta\tau_{m+1} > n-k) \cdot \mathsf{P}(Y > x) \\
&\le\ c^* C n \sum_{k=1}^{n} \mathsf{P}\left(\sum_{i=1}^{m} \Delta\tau_i = k\right) \mathsf{P}(\Delta\tau_{m+1} > n-k) \cdot \mathsf{P}(Y > x) \\
&=\ c^* C n \mathsf{P}(N(n) = m) \cdot \mathsf{P}(Y > x).
\end{aligned}$$

B.5 Proof of Lemma A.1

We start with a basic result, Proposition B.1 below, which is used to prove Lemma A.1 and some other ones. The proof of this proposition is given in Appendix B.17.

Proposition B.1 *For any $\gamma > 0$ and $x > y \ge 0$,*

$$(x+y)^\gamma \le x^\gamma + C\left(1 - \frac{y}{x}\right)^{-1} y x^{\gamma-1}, \tag{B.9}$$

$$(x-y)^\gamma \ge x^\gamma - C\left(1 - \frac{y}{x}\right)^{-1} y x^{\gamma-1}, \tag{B.10}$$

where C is independent of x and y.

In what follows, we give the proof of Lemma A.1. We first prove the statement (i). It follows from $X^\theta \in \mathcal{L}$ that $\lim_{y\to\infty} e^{\varepsilon y}\mathsf{P}(X^\theta > y) = \infty$ for any $\varepsilon > 0$. Thus letting $x = y^{1/\theta}$ for $y > 0$, we have

$$\lim_{x\to\infty} e^{\varepsilon x^\theta}\mathsf{P}(X > x) = \lim_{x\to\infty} e^{\varepsilon x^\theta}\mathsf{P}(X^\theta > x^\theta) = \lim_{y\to\infty} e^{\varepsilon y}\mathsf{P}(X^\theta > y) = \infty.$$

Next, we prove the statement (ii). Fix y ($0 \le y < x/2$) arbitrarily. We then have

$$1 \ge \frac{\mathsf{P}(X^\eta > x+y)}{\mathsf{P}(X^\eta > x)} = \frac{\mathsf{P}(X^\theta > (x+y)^{\theta/\eta})}{\mathsf{P}(X^\theta > x^{\theta/\eta})}. \tag{B.11}$$

It follows from $0 < \theta/\eta < 1$ and Proposition B.1 that for any $x \ge 1$ and $0 \le y < x/2$,

$$(x+y)^{\theta/\eta} \le x^{\theta/\eta} + Cyx^{\theta/\eta-1} \le x^{\theta/\eta} + Cy,$$

from which and $X^\theta \in \mathcal{L}$ we obtain

$$\lim_{x\to\infty} \frac{\mathsf{P}(X^\theta > (x+y)^{\theta/\eta})}{\mathsf{P}(X^\theta > x^{\theta/\eta})} \ge \lim_{x\to\infty} \frac{\mathsf{P}(X^\theta > x^{\theta/\eta} + Cy)}{\mathsf{P}(X^\theta > x^{\theta/\eta})} = 1. \tag{B.12}$$

Combining (B.12) with (B.11) yields $\mathsf{P}(X^\eta > x+y) \overset{x}{\sim} \mathsf{P}(X^\eta > x)$ for any $y > 0$, i.e., $X^\eta \in \mathcal{L}$.

B.6 Proof of Lemma A.2

(If part) Proposition B.1 implies that $(x+y)^{1/\theta} \le x^{1/\theta} + Cyx^{1/\theta-1}$ for any $x>0$ and $0 \le y < x/2$. Thus we have for any $y>0$,

$$1 \ge \lim_{x\to\infty} \frac{\mathsf{P}(X^\theta > x+y)}{\mathsf{P}(X^\theta > x)} \ge \lim_{x\to\infty} \frac{\mathsf{P}(X > x^{1/\theta} + Cy\cdot(x^{1/\theta})^{1-\theta})}{\mathsf{P}(X > x^{1/\theta})} = 1,$$

which shows that $X^\theta \in \mathcal{L}$.

(Only-if part) We fix ξ such that $x^\theta > 2|\xi|$. It then follows from Proposition B.1 that

$$\begin{aligned} (x-\xi x^{1-\theta})^\theta &\ge x^\theta - C\left(1-\frac{\xi}{x^\theta}\right)^{-1}\xi \ge x^\theta - 2C\xi, && \xi \ge 0,\\ (x-\xi x^{1-\theta})^\theta &\le x^\theta + C\left(1-\frac{-\xi}{x^\theta}\right)^{-1}(-\xi) \le x^\theta + 2C(-\xi), && \xi < 0. \end{aligned}$$

Thus for $\xi \ge 0$,

$$1 \le \lim_{x\to\infty} \frac{\mathsf{P}(X > x-\xi x^{1-\theta})}{\mathsf{P}(X > x)} \le \lim_{x\to\infty} \frac{\mathsf{P}(X^\theta > x^\theta - C\xi)}{\mathsf{P}(X^\theta > x^\theta)} = 1,$$

where the last equality follows from $X^\theta \in \mathcal{L}$. Similarly, for $\xi < 0$,

$$1 \ge \lim_{x\to\infty} \frac{\mathsf{P}(X > x-\xi x^{1-\theta})}{\mathsf{P}(X > x)} \ge \lim_{x\to\infty} \frac{\mathsf{P}(X^\theta > x^\theta + C(-\xi))}{\mathsf{P}(X^\theta > x^\theta)} = 1.$$

As a result, $\mathsf{P}(X > x-\xi x^{1-\theta}) \overset{x}{\sim} \mathsf{P}(X > x)$ for any $\xi \in \mathbb{R}$.

B.7 Proof of Lemma A.3

Proposition B.1 implies that for all $x > 2$,

$$\begin{aligned} (x+1)^{1/\theta} &\le x^{1/\theta} + C\cdot(x^{1/\theta})^{1-\theta},\\ (x-1)^{1/\theta} &\ge x^{1/\theta} - C\cdot(x^{1/\theta})^{1-\theta}. \end{aligned} \tag{B.13}$$

Using (B.13), we have

$$\begin{aligned} 1 \ge \lim_{x\to\infty} \frac{\mathsf{P}(X^\theta > x+1)}{\mathsf{P}(X^\theta > x)} &\ge \lim_{x\to\infty} \frac{\mathsf{P}(X > x^{1/\theta} + C\cdot(x^{1/\theta})^{1-\theta})}{\mathsf{P}(X > x^{1/\theta})}\\ &= \lim_{x\to\infty} \frac{\mathsf{P}(X > x + Cx^{1-\theta})}{\mathsf{P}(X > x)}. \end{aligned}$$

Similarly, we have

$$1 \ge \lim_{x\to\infty} \frac{\mathsf{P}(X^\theta > x)}{\mathsf{P}(X^\theta > x-1)} \ge \lim_{x\to\infty} \frac{\mathsf{P}(X > x)}{\mathsf{P}(X > x - Cx^{1-\theta})}.$$

Therefore, if either

$$\mathsf{P}(X > x + Cx^{1-\theta}) \overset{x}{\sim} \mathsf{P}(X > x) \text{ or } \mathsf{P}(X > x - Cx^{1-\theta}) \overset{x}{\sim} \mathsf{P}(X > x), \tag{B.14}$$

then $\mathsf{P}(X^\theta > x + 1) \overset{x}{\sim} \mathsf{P}(X^\theta > x)$, which shows $X^\theta \in \mathcal{L}$, i.e., $X \in \mathcal{L}^{1/\theta}$.

In what follows, we prove that either of the two limits in (B.14) holds. By assumption, there exists some $\xi \in \mathbb{R}\backslash\{0\}$ such that

$$\mathsf{P}(X > h_\xi(x)) \overset{x}{\sim} \mathsf{P}(X > x), \tag{B.15}$$

where $h_\xi(x) = x - \xi x^{1-\theta}$ for $x \ge 0$. Thus since $h_\xi(x) \overset{x}{\sim} x$,

$$\lim_{x\to\infty} \frac{\mathsf{P}(X > h_\xi(h_\xi(x)))}{\mathsf{P}(X > x)} = \lim_{x\to\infty} \frac{\mathsf{P}(X > h_\xi(h_\xi(x)))}{\mathsf{P}(X > h_\xi(x))} \lim_{x\to\infty} \frac{\mathsf{P}(X > h_\xi(x))}{\mathsf{P}(X > x)} = 1. \tag{B.16}$$

Further, for any $0 < \varepsilon < 1$, the following inequalities are satisfied for all sufficiently large $x > 0$.

$$h_\xi(x) \ge x - (1+\varepsilon)\xi x^{1-\theta} \ge h_\xi(h_\xi(x)), \qquad \text{if } \xi > 0. \tag{B.17}$$

$$h_\xi(x) \le x - (1+\varepsilon)\xi x^{1-\theta} \le h_\xi(h_\xi(x)), \qquad \text{if } \xi < 0. \tag{B.18}$$

The second inequalities in (B.17) and (B.18) hold because

$$\left[x - (1+\varepsilon)\xi x^{1-\theta}\right] - h_\xi(h_\xi(x)) = \xi\left[-\varepsilon x^{1-\theta} + \{h_\xi(x)\}^{1-\theta}\right] \overset{x}{\sim} \xi(1-\varepsilon)x^{1-\theta}.$$

It follows from (B.15)–(B.18) and the squeeze theorem (see, e.g. [26, Chapter IV, Theorem 14.3]) that for $\xi \in \mathbb{R} \setminus \{0\}$ such that (B.15) holds,

$$\mathsf{P}(X > x - (1+\varepsilon)\cdot \xi x^{1-\theta}) \overset{x}{\sim} \mathsf{P}(X > x), \qquad 0 < \varepsilon < 1,$$

which implies that

$$\mathsf{P}(X > x - (1+\varepsilon)^n \cdot \xi x^{1-\theta}) \overset{x}{\sim} \mathsf{P}(X > x), \qquad \text{for all } n = 1, 2, \dots.$$

As a result, either of the two limits in (B.14) holds.

B.8 Proof of Lemma A.4

The case of $u = 0$ is obvious. Therefore, we focus on the case of $u > 0$. For any $x \ge u$, we have

$$\frac{\mathsf{P}(X > x - u)}{\mathsf{P}(X > x)} = \frac{\mathsf{P}(X^\theta > (x-u)^\theta)}{\mathsf{P}(X^\theta > x^\theta)} \le \frac{\mathsf{P}(X^\theta > x^\theta - u^\theta)}{\mathsf{P}(X^\theta > x^\theta)}, \tag{B.19}$$

where we use $(x-u)^\theta \ge x^\theta - u^\theta$ for $0 \le u \le x$. Let y denote a nonnegative number such that $y = x^\theta - u^\theta$. We then have

$$\frac{\mathsf{P}(X^\theta > x^\theta - u^\theta)}{\mathsf{P}(X^\theta > x^\theta)} = \frac{\mathsf{P}(X^\theta > y)}{\mathsf{P}(X^\theta > y + u^\theta)} = \prod_{i=0}^{\lceil u^\theta \rceil - 1} \frac{\mathsf{P}\left(X^\theta > y + i\dfrac{u^\theta}{\lceil u^\theta \rceil}\right)}{\mathsf{P}\left(X^\theta > y + (i+1)\dfrac{u^\theta}{\lceil u^\theta \rceil}\right)}. \tag{B.20}$$

It follows from $X^\theta \in \mathcal{L}$ that for any $\varepsilon > 0$ there exists some $\breve{y}_\varepsilon > 0$ such that for all $y > \breve{y}_\varepsilon$,

$$\frac{\mathsf{P}(X^\theta > y)}{\mathsf{P}(X^\theta > y + \gamma)} \le e^\varepsilon, \qquad 0 \le \gamma \le 1,$$

where $\breve{y}_\varepsilon$ is independent of γ. Thus since $0 < u^\theta / \lceil u^\theta \rceil \le 1$, we have

$$\prod_{i=0}^{\lceil u^\theta \rceil - 1} \frac{\mathsf{P}\left(X^\theta > y + i\dfrac{u^\theta}{\lceil u^\theta \rceil}\right)}{\mathsf{P}\left(X^\theta > y + (i+1)\dfrac{u^\theta}{\lceil u^\theta \rceil}\right)} \le e^{\varepsilon \lceil u^\theta \rceil} \le e^{\varepsilon(u^\theta + 1)}, \qquad y > \breve{y}_\varepsilon,$$

from which, (B.19) and (B.20) it follows that

$$\frac{\mathsf{P}(X > x - u)}{\mathsf{P}(X > x)} \le e^{\varepsilon(u^\theta + 1)}, \tag{B.21}$$

for all $x, u \ge 0$ such that $x^\theta - u^\theta > \breve{y}_\varepsilon$.

Note here that for all $0 < u \le g(x)$,

$$\liminf_{x\to\infty}(x^\theta - u^\theta) \ge \liminf_{x\to\infty}\left[x^\theta - \{g(x)\}^\theta\right] = \liminf_{x\to\infty} x^\theta \left[1 - \left(\frac{g(x)}{x}\right)^\theta\right] = \infty,$$

where the last equality follows from $\limsup_{x\to\infty} g(x)/x < 1$. As a result, there exists some $\breve{x}_\varepsilon > 0$ such that for all $x > \breve{x}_\varepsilon$ and $0 < u \le g(x)$, we have $x^\theta - u^\theta > \breve{y}_\varepsilon$ and thus (B.21) holds.

B.9 Proof of Lemma A.5

Suppose $X \in \mathcal{C}$. The definition of $\mathcal{C}$ shows that for any $v > 1$ there exists some $c(v) > 0$ such that $\lim_{v\downarrow 1} c(v) = 1$ and

$$\liminf_{x\to\infty} \frac{\mathsf{P}(X > vx)}{\mathsf{P}(X > x)} = c(v).$$

Since $x + 1 \le vx$ for all sufficiently large $x > 0$, we have for any $0 < \theta \le 1$,

$$\liminf_{x\to\infty} \frac{\mathsf{P}(X^\theta > x + 1)}{\mathsf{P}(X^\theta > x)} \ge \liminf_{x\to\infty} \frac{\mathsf{P}(X > (vx)^{1/\theta})}{\mathsf{P}(X > x^{1/\theta})} = c(v^{1/\theta}) \to 1, \quad \text{as } v \downarrow 1.$$

On the other hand, it is clear that $\mathsf{P}(X^\theta > x + 1) \lesssim_x \mathsf{P}(X^\theta > x)$. Therefore, we obtain $\mathsf{P}(X^\theta > x + 1) \stackrel{x}{\sim} \mathsf{P}(X^\theta > x)$, i.e., $X^\theta \in \mathcal{L}$.

B.10 Proof of Lemma A.6

For any $0 < \beta \le 1$, it follows from (A.1) that for all sufficiently large $x > 0$,

$$\begin{aligned} 1 \le \frac{\mathsf{P}(X > x - x^{1-\beta})}{\mathsf{P}(X > x)} &= \exp\{Q_X(x) - Q_X(x - x^{1-\beta})\} \\ &\le \exp\{\alpha Q_X(x)/x^{\beta}\}. \end{aligned} \tag{B.22}$$

Since $Q_X \in \mathcal{SC}_\alpha$,

$$Q_X(x) \le Cx^\alpha, \quad \text{for all } x \ge x_0.$$

Thus for any $\beta \in (\alpha, 1]$, we have

$$1 \le \frac{\mathsf{P}(X > x - x^{1-\beta})}{\mathsf{P}(X > x)} \le \exp\{Cx^{\alpha-\beta}\} \to 1, \ \text{ as } x \to \infty,$$

which implies $X^\beta \in \mathcal{L}$ due to Lemma A.3. Further, if (A.3) holds, then substituting (A.3) into (B.22) with $\beta = \alpha$ yields $\mathsf{P}(X > x - x^{1-\alpha}) \overset{x}{\sim} \mathsf{P}(X > x)$, i.e., $X^\alpha \in \mathcal{L}$.

B.11 Proof of Lemma A.7

It follows from that Lemma 4.4 in [14] that there exists some nonincreasing slowly varying function l_0 such that $l_0(0) = 1$, $\lim_{x\to\infty} l_0(x) = 0$ and $\mathsf{P}(U > x) = o(l_0(x)\mathsf{P}(Y > x))$. Thus for any $\varepsilon > 0$ and $x_0 \ge 0$, there exists some $x_2 := x_2(\varepsilon, x_0) > x_0$ such that

$$\overline{F}_U(x) < \varepsilon l_0(x)\overline{F}_Y(x) \le \overline{F}_U(0), \quad \forall x \ge x_2.$$

Let $x_1 := x_1(\varepsilon)$ denote

$$x_1 = \inf\left\{x \in [x_0, x_2]; \overline{F}_U(x) \le \varepsilon l_0(x_2)\overline{F}_Y(x_2)\right\}.$$

Note here that since $\overline{F}_U$ is right-continuous,

$$\overline{F}_U(x_1) \le \varepsilon l_0(x_2)\overline{F}_Y(x_2).$$

We then define $Z(\varepsilon, x_0)$ as an r.v. in $\mathbb{R}$ such that (see Figure 1)

$$\overline{F}_{Z(\varepsilon,x_0)}(x) = \begin{cases} \overline{F}_U(x), & x < 0, \\ \overline{F}_U(0), & 0 \le x < x_0, \\ \overline{F}_U(x), & x_0 \le x < x_1, \\ \varepsilon l_0(x_2)\overline{F}_Y(x_2), & x_1 \le x < x_2, \\ \varepsilon l_0(x)\overline{F}_Y(x), & x \ge x_2. \end{cases} \tag{B.23}$$

Clearly, $\overline{F}_{Z(\varepsilon,x_0)}(x) \ge \overline{F}_U(x)$ for all $x \in \mathbb{R}$. It thus follows from (B.23) that

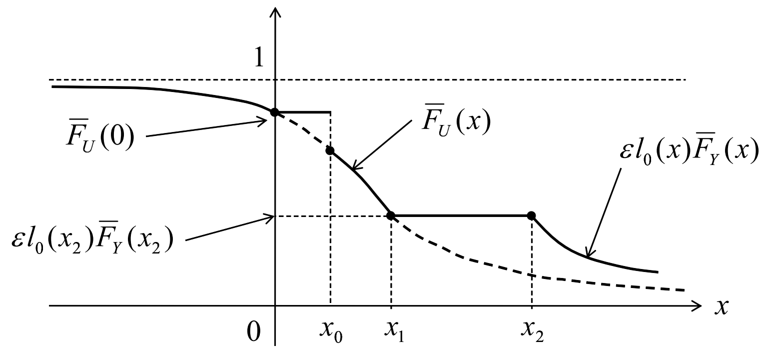

Figure 1: Tail distribution of $Z(\varepsilon, x_0)$.

$$\begin{aligned}0 &\le \mathsf{E}[Z(\varepsilon, x_0)] - \mathsf{E}[U] \\ &=: S_1(\varepsilon, x_0) + S_2(\varepsilon, x_0) + S_3(\varepsilon, x_0),\end{aligned} \tag{B.24}$$

where

$$S_1(\varepsilon, x_0) = \int_0^{x_0} (\overline{F}_U(0) - \overline{F}_U(x))\mathrm{d}x, \tag{B.25}$$

$$S_2(\varepsilon, x_0) = \int_{x_1}^{x_2} \big(\varepsilon l_0(x_2)\overline{F}_Y(x_2) - \overline{F}_U(x)\big)\, \mathrm{d}x, \tag{B.26}$$

$$S_3(\varepsilon, x_0) = \int_{x_2}^{\infty} \big(\varepsilon l_0(x)\overline{F}_Y(x) - \overline{F}_U(x)\big)\, \mathrm{d}x. \tag{B.27}$$

From (B.24) and (B.25), we have

$$\lim_{x_0\to\infty} \sum_{j=1}^{3} S_j(\varepsilon, x_0) \ge \lim_{x_0\to\infty} S_1(\varepsilon, x_0) = \infty. \tag{B.28}$$

Further, from (B.26) and (B.27), we have

$$\begin{aligned}S_2(\varepsilon, x_0) + S_3(\varepsilon, x_0) &\le \int_{x_1}^{\infty} \big(\varepsilon l_0(x)\overline{F}_Y(x) - \overline{F}_U(x)\big)\, \mathrm{d}x \\ &\le \int_{x_1}^{\infty} \varepsilon\overline{F}_Y(x)\mathrm{d}x,\end{aligned}$$

where the second inequality follows from $l_0(x) \le 1$ for $x \ge 0$. Therefore, since $\mathsf{E}[Y] < \infty$,

$$\lim_{\varepsilon\downarrow 0}(S_2(\varepsilon, x_0) + S_3(\varepsilon, x_0)) = 0,$$

which leads to

$$\lim_{x_0\downarrow 0}\lim_{\varepsilon\downarrow 0} \sum_{j=1}^{3} S_j(\varepsilon, x_0) = \lim_{x_0\downarrow 0}\lim_{\varepsilon\downarrow 0} S_1(\varepsilon, x_0) = 0. \tag{B.29}$$

Note here that $\sum_{j=1}^{3} S_j(\varepsilon, x_0)$ is continuous for all $\varepsilon > 0$ and $x_0 > 0$. Thus, (B.24), (B.28) and (B.29) imply that $\mathsf{E}[Z(\varepsilon, x_0)] - \mathsf{E}[U]$ can take any value in $(0, \infty)$. As a result, the statement of Lemma A.7 holds for $Z = Z(\varepsilon, x_0)$.

B.12 Proof of Lemma A.8

Event $\{N_X(x) > u + x/\mathsf{E}[X]\}$ implies event $\{\sum_{n=1}^{\lfloor u+x/\mathsf{E}[X]\rfloor} X_n \le x\}$ and thus

$$\begin{aligned}\mathsf{P}\left(N_X(x) - \frac{x}{\mathsf{E}[X]} > u\right) &\le \mathsf{P}\left(\sum_{n=1}^{\lfloor u+x/\mathsf{E}[X]\rfloor} X_n \le x\right) \\ &\le \mathsf{P}\left(\sum_{n=1}^{\lfloor u+x/\mathsf{E}[X]\rfloor} (1-\widetilde{X}_n) \ge u-1\right), \end{aligned} \tag{B.30}$$

where the $\widetilde{X}_n$'s ($n = 1, 2, \dots$) are independent copies of $\widetilde{X} := X/\mathsf{E}[X]$. Using Markov's inequality (see, e.g., [44]), we have for any $s > 0$,

$$\begin{aligned}\mathsf{P}\left(\sum_{n=1}^{\lfloor u+x/\mathsf{E}[X]\rfloor} (1-\widetilde{X}_n) \ge u-1\right) &\le e^{-s(u-1)}\left(\mathsf{E}[e^{s(1-\widetilde{X})}]\right)^{\lfloor u+x/\mathsf{E}[X]\rfloor} \\ &\le e^{s(1+x/\mathsf{E}[X])}\left(\mathsf{E}[e^{-s\widetilde{X}}]\right)^{\lfloor u+x/\mathsf{E}[X]\rfloor} \\ &\le e^{s(1+x/\mathsf{E}[X])}\left(\mathsf{E}[e^{-s\widetilde{X}}]\right)^{u+x/\mathsf{E}[X]}, \end{aligned} \tag{B.31}$$

where the last inequality follows from $\mathsf{E}[e^{s(1-\widetilde{X})}] \ge \exp\{s(1-\mathsf{E}[\widetilde{X}])\} = 1$ due to Jensen's inequality (see, e.g., [44]). Further, for any $s > 0$,

$$\mathsf{E}[e^{-s\widetilde{X}}] \le 1 - s\mathsf{E}[\widetilde{X}] + s^2\mathsf{E}[\widetilde{X}^2] = 1 - s + s^2\mathsf{E}[\widetilde{X}^2], \tag{B.32}$$

because $e^{-x} \le 1 - x + x^2$ ($\forall x \ge 0$) and $\widetilde{X} \ge 0$ w.p.1. Substituting (B.32) into (B.31), we obtain

$$\begin{aligned}\mathsf{P}\left(\sum_{n=1}^{\lfloor u+x/\mathsf{E}[X]\rfloor} (1-\widetilde{X}_n) \ge u-1\right) &\le e^{s(1+x/\mathsf{E}[X])}\left(1 - s + s^2\mathsf{E}[\widetilde{X}^2]\right)^{u+x/\mathsf{E}[X]} \\ &\le e^{s(1+x/\mathsf{E}[X])} e^{(-s+s^2\mathsf{E}[\widetilde{X}^2])(u+x/\mathsf{E}[X])} \\ &\le e^s \exp\left\{-su + s^2\mathsf{E}[\widetilde{X}^2](\delta + 1/\mathsf{E}[X])x\right\} \\ &=: e^s \exp\left\{-su + s^2\widehat{C}(\delta)x\right\}, \end{aligned} \tag{B.33}$$

where we use $1 + x \le e^x$ ($x \in \mathbb{R}$) and $u \le \delta x$ in the second and third inequalities.

We now fix $s = (u/x)\{2\widehat{C}(\delta)\}^{-1}$, which is finite for any fixed $\delta > 0$ due to $0 \le u \le \delta x$. Thus (B.33) yields

$$\mathsf{P}\left(\sum_{n=1}^{\lfloor u+x/\mathsf{E}[X]\rfloor} (1-\widetilde{X}_n) \ge u-1\right) \le \exp\left\{\frac{\delta}{2\widehat{C}(\delta)}\right\} \cdot \exp\left\{-\frac{1}{4\widehat{C}(\delta)}\frac{u^2}{x}\right\}.$$

Combining this and (B.30), we obtain (A.4).

B.13 Proof of Lemma A.9

For all $n = 1, 2, \dots$ and $k = 1, 2, \dots, n$,

$$\left\{ \max_{1 \le k \le n} \sum_{i=1}^{k} U_i \ge x \right\} = \bigcup_{1 \le k \le n} \left\{ \sum_{i=1}^{k} U_i \ge x \right\}, \quad x > 0,$$

$$\left\{ \sum_{i=1}^{k} U_i \ge x \right\} \subseteq \bigcup_{1 \le i \le k} \{U_i \ge x/k\} \subseteq \bigcup_{1 \le i \le k} \{U_i \ge x/n\}, \quad x > 0.$$

Thus for any fixed positive integer n_0 and all $n = 1, 2, \dots, n_0$, we have

$$\left\{ \max_{1 \le k \le n} \sum_{i=1}^{k} U_i \ge x \right\} \subseteq \bigcup_{1 \le i \le n} \{U_i \ge x/n\} \subseteq \bigcup_{1 \le i \le n} \{U_i \ge x/n_0\}, \quad x > 0,$$

which leads to

$$\mathsf{P}\left(\max_{1 \le k \le n} \sum_{i=1}^{k} U_i \ge x \right) \le n\mathsf{P}(U \ge x/n_0), \quad n = 1, 2, \dots, n_0,\ x > 0.$$

Further, it follows from $U^+ \in \mathcal{C} \subset \mathcal{D}$ that $\mathsf{P}(U \ge x/n_0) \le C\mathsf{P}(U \ge x)$ for all $x > 0$ and thus

$$\mathsf{P}\left(\max_{1 \le k \le n} \sum_{i=1}^{k} U_i \ge x \right) \le Cn\mathsf{P}(U \ge x), \quad n = 1, 2, \dots, n_0,\ x > 0.$$

Therefore, it suffices to show that (A.6) holds for all sufficiently large n.

Let $\widetilde{U}_i = \min(U_i, vx)$ for $i = 1, 2, \dots$, where $0 < v < 1$ is a constant. Then $\mathsf{E}[\widetilde{U}_1] \le 0$ because $\mathsf{E}[U] = 0$. Thus for all $x > 0$,

$$\begin{aligned}
&\mathsf{P}\left(\max_{1 \le k \le n} \sum_{i=1}^{k} U_i \ge x \right) \\
&\quad \le \mathsf{P}\left(\max_{1 \le i \le n} U_i > vx \right) + \mathsf{P}\left(\max_{1 \le k \le n} \sum_{i=1}^{k} U_i \ge x, \max_{1 \le i \le n} U_i \le vx \right) \\
&\quad \le n\mathsf{P}(U > vx) + \mathsf{P}\left(\max_{1 \le k \le n} \sum_{i=1}^{k} \widetilde{U}_i \ge x \right) \\
&\quad \le Cn\mathsf{P}(U > x) + \mathsf{P}\left(\max_{1 \le k \le n} \sum_{i=1}^{k} W_i \ge x \right),
\end{aligned} \tag{B.34}$$

where $W_i = \widetilde{U}_i - \mathsf{E}[\widetilde{U}_1]$ for $i = 1, 2, \dots$. The second term on the right hand side of (B.34) is estimated as follows. For any $s > 0$,

$$\mathsf{P}\left(\max_{1 \le k \le n} \sum_{i=1}^{k} W_i \ge x \right) = \mathsf{P}\left(\max_{1 \le k \le n} \exp\left\{ s \sum_{i=1}^{k} W_i \right\} \ge e^{sx} \right).$$

Since function $x \mapsto e^{sx}$ is convex, $\{\exp\{s\sum_{i=1}^k W_i\}; k = 1, 2, \dots\}$ is submartingale (see, e.g., [44]). It thus follows from Doob's submartingale inequality (see, e.g., [44]) that for any $s > 0$,

$$\begin{aligned}
\mathsf{P}\left(\max_{1\le k\le n}\sum_{i=1}^k W_i \ge x\right) &\le e^{-sx}\mathsf{E}\left[\exp\left\{s\sum_{i=1}^n W_i\right\}\right] \\
&= e^{-sx}\mathsf{E}[\exp\{s\widetilde{U}_1\}]^n \cdot \exp\{-sn\mathsf{E}[\widetilde{U}_1]\}. \qquad \text{(B.35)}
\end{aligned}$$

Fix $1 < q < \min(r, 2)$ and

$$s = \frac{1}{vx}\log\left(\frac{v^{q-1}x^q}{n\mathsf{E}[(U^+)^q]} + 1\right). \qquad \text{(B.36)}$$

Then following the proof of Lemma 2.3 in [43], we can show that there exist positive constants $\widetilde{C}_1 := \widetilde{C}_1(v, \gamma)$ and $v := v(r, p) < \min(r-1, 1)/(2p)$ such that for all $x \ge \gamma n$ and $n = 1, 2, \dots$,

$$e^{-sx}\mathsf{E}[\exp\{s\widetilde{U}_1\}]^n \le \widetilde{C}_1 x^{-p}.$$

Therefore, it remains to estimate $\exp\{-sn\mathsf{E}[\widetilde{U}_1]\}$ in (B.35).

From (B.36), $\mathsf{E}[\widetilde{U}_1] \le 0$, $x \ge \gamma n$ and $n \ge 1$, we have

$$-\, sn\mathsf{E}[\widetilde{U}_1] \le \frac{1}{v\gamma}\log\left(\frac{v^{q-1}x^q}{\mathsf{E}[(U^+)^q]} + 1\right)(-\mathsf{E}[\widetilde{U}_1]). \qquad \text{(B.37)}$$

Further, it follows from that $\mathsf{E}[U] = 0$ and $\mathsf{P}(U^+ > x) = o(x^{-r})$ (due to $\mathsf{E}[(U^+)^r] < \infty$) that for all $x > 0$,

$$\begin{aligned}
-\mathsf{E}[\widetilde{U}_1] &= \mathsf{E}[U \cdot 1\!\!1(U > vx)] \\
&= \mathsf{E}[U^+ \cdot 1\!\!1(U^+ > vx)] \\
&= vx\mathsf{P}(U^+ > vx) + \int_{vx}^\infty \mathsf{P}(U^+ > y)\mathrm{d}y \\
&= o(x^{-r+1}).
\end{aligned}$$

This equation and (B.37) imply that for all $x \ge \gamma n$ and $n = 1, 2, \dots$,

$$\exp\{-sn\mathsf{E}[\widetilde{U}_1]\} \le \widetilde{C}_2 < \infty,$$

where

$$\widetilde{C}_2 := \widetilde{C}_2(v, \gamma) = \sup_{x\ge\gamma}\exp\left\{\frac{1}{v\gamma}\log\left(\frac{v^{q-1}x^q}{\mathsf{E}[(U^+)^q]} + 1\right)Cx^{-r+1}\right\}.$$

As a result, letting $\widetilde{C} := \widetilde{C}(v, \gamma) = \widetilde{C}_1(v, \gamma)\widetilde{C}_2(v, \gamma)$, we can see that the inequality (A.5) holds for all $x \ge \gamma n$ and $n = 1, 2, \dots$.

B.14 Proof of Lemma A.10

Let $U' = U - \mathsf{E}[U] - \varepsilon$ and $U'_i = U_i - \mathsf{E}[U_i] - \varepsilon$ for $i = 1, 2, \dots$, where $\varepsilon > 0$. Clearly, $\mathsf{E}[U'] = -\varepsilon < 0$. It follows from the theorem in [28] that for all $x \ge (\mathsf{E}[U] + \varepsilon)n$ and $n = 1, 2, \dots$,

$$\begin{aligned}
\mathsf{P}\left(\max_{1\le k\le n}\sum_{i=1}^{k} U_i \ge x\right) &\le \mathsf{P}\left(\max_{1\le k\le n}\sum_{i=1}^{k} U'_i \ge x - (\mathsf{E}[U]+\varepsilon)n\right) \\
&\le \frac{C_1}{\varepsilon}\int_{x-(\mathsf{E}[U]+\varepsilon)n}^{x-\mathsf{E}[U]n} \mathsf{P}(U' > y)\mathrm{d}y \\
&\le C_1 n \mathsf{P}(U' > x - (\mathsf{E}[U]+\varepsilon)n) \\
&\le C_1 n \mathsf{P}(U > x - (\mathsf{E}[U]+\varepsilon)n),
\end{aligned} \tag{B.38}$$

where $C_1 > 0$ is some constant independent of x and n. Since $U^+ \in \mathcal{C} \subset \mathcal{D}$, we have for all $x \ge (1+\delta)(\mathsf{E}[U]+\varepsilon)n$, $n = 1, 2, \dots$ and $\delta > 0$,

$$\mathsf{P}(U > x - (\mathsf{E}[U]+\varepsilon)n) \le \mathsf{P}(U > \delta x/(1+\delta)) \le C_2 \mathsf{P}(U > x), \tag{B.39}$$

where $C_2 > 0$ is some constant that depends on δ and $(1+\delta)(\mathsf{E}[U]+\varepsilon)$. Substituting (B.39) into (B.38), we have

$$\mathsf{P}\left(\max_{1\le k\le n}\sum_{i=1}^{k} U_i \ge x\right) \le C_1 C_2 n \mathsf{P}(U > x), \quad x \ge (1+\delta)(\mathsf{E}[U]+\varepsilon)n,\ n = 1, 2, \dots .$$

B.15 Proof of Lemma A.11

It follows from (A.7) that for any $\varepsilon > 0$ there exists some $x_0 > 0$ such that for all $x > x_0$,

$$\begin{aligned}
\overline{F}_{i,j}(x) &\le (\widetilde{F}_{i,j} + \varepsilon)\mathsf{P}(Y > x), && 1 \le i \le m_0,\ 1 \le j \le m_1, \\
\overline{G}_{i,j}(x) &\le (\widetilde{G}_{i,j} + \varepsilon)\mathsf{P}(Y > x), && 1 \le i \le m_1,\ 1 \le j \le m_2.
\end{aligned}$$

We now define $\boldsymbol{P} = (P_{i,j})$ and $\boldsymbol{Q} = (Q_{i,j})$ as $m_0 \times m_1$ and $m_1 \times m_2$ matrix-valued functions on $\mathbb{R}$ such that

$$\overline{P}_{i,j}(x) := P_{i,j}(\infty) - P_{i,j}(x) = \begin{cases} F_{i,j}(\infty), & x \le 0, \\ \overline{F}_{i,j}(x), & 0 < x \le x_0, \\ (\widetilde{F}_{i,j} + \varepsilon)\mathsf{P}(Y > x), & x > x_0, \end{cases}$$

$$\overline{Q}_{i,j}(x) := Q_{i,j}(\infty) - Q_{i,j}(x) = \begin{cases} G_{i,j}(\infty), & x \le 0, \\ \overline{G}_{i,j}(x), & 0 < x \le x_0, \\ (\widetilde{G}_{i,j} + \varepsilon)\mathsf{P}(Y > x), & x > x_0. \end{cases}$$

Clearly, $\overline{F}_{i,j}(x) \le \overline{P}_{i,j}(x)$ and $\overline{G}_{i,j}(x) \le \overline{Q}_{i,j}(x)$ for all $x \in \mathbb{R}$. Thus, following the proof of Proposition A.3 in [33] and letting $\varepsilon \downarrow 0$, we can readily prove that

$$\limsup_{x\to\infty} \frac{\overline{\boldsymbol{F} * \boldsymbol{G}}(x)}{\mathsf{P}(Y > x)} \le \lim_{x\to\infty} \frac{\overline{\boldsymbol{P} * \boldsymbol{Q}}(x)}{\mathsf{P}(Y > x)} = \widetilde{\boldsymbol{F}}\boldsymbol{G}(\infty) + \boldsymbol{F}(\infty)\widetilde{\boldsymbol{G}},$$

which implies that the first statement (A.8) is true. The second statement (A.9) follows from the first statement (A.8). As for the third statement (A.10), it is straightforward from Lemma 6 in [21] because $\overline{\boldsymbol{F}^{*n}}(x) \le \overline{\boldsymbol{P}^{*n}}(x)$ for all $x \in \mathbb{R}$.

B.16 Proof of Lemma B.1

It follows from (2.1) that there exists some $x_* > 0$ such that

$$Q(x/3) \ge \frac{1}{3} Q(x), \qquad \text{for all } x \ge x_*. \tag{B.40}$$

Let η denote any fixed positive number such that $\eta x_*^2 \ge 1$. We then discuss three cases: (a) $0 \le x < \eta x_*^2$, (b) $x > \eta u^2$ and (c) $\eta x_*^2 \le x \le \eta u^2$ separately. In case (a), (B.1) holds for $C \ge e^{(\eta x_*)^2}$ because $Ce^{-\eta x} > Ce^{-(\eta x_*)^2} \ge 1$. In case (b), (B.1) also holds for $C \ge e$ because $Ce^{-\eta u^2/x} > Ce^{-1} \ge 1$. Therefore in what follows, we consider case (c).

For all $t \ge 0$,

$$\begin{aligned}
\breve{B}(t) - \breve{b}t &\le \left[\Delta \breve{B}_0^* + \Delta \breve{B}_{N(t-\tau_0)+1}^* + \sum_{i=1}^{N(t-\tau_0)} \Delta \breve{B}_i \right] \\
&\qquad - \min(\breve{b}, 0)(\Delta\tau_0 + \Delta\tau_{N(t-\tau_0)+1}) - \breve{b} \sum_{i=1}^{N(t-\tau_0)} \Delta\tau_i \\
&= \Delta\Omega_0 + \Delta\Omega_{N(t-\tau_0)+1} + \sum_{i=1}^{N(t-\tau_0)} (\Delta\breve{B}_i - \breve{b}\Delta\tau_i),
\end{aligned}$$

where $N(t) = \max\{n \ge 0; \sum_{i=1}^{n} \Delta\tau_i \le t\}$ for $t \ge 0$. Thus we obtain

$$\begin{aligned}
&\mathsf{P}\left(\sup_{0 \le t \le x} \{\breve{B}(t) - \breve{b}t\} > u \right) \\
&\le \mathsf{P}\left(\Delta\Omega_0 > \frac{u}{3} \right) + \mathsf{P}\left(\Delta\Omega_1 > \frac{u}{3} \right) + \mathsf{P}\left(\max_{1 \le n \le N(x-\tau_0)} \sum_{i=1}^{n} (\Delta\breve{B}_i - \breve{b}\Delta\tau_i) > \frac{u}{3} \right) \\
&\le \mathsf{P}\left(\Delta\Omega_0 > \frac{u}{3} \right) + \mathsf{P}\left(\Delta\Omega_1 > \frac{u}{3} \right) + \mathsf{P}\left(\max_{1 \le n \le N(x)} \sum_{i=1}^{n} (\Delta\breve{B}_i - \breve{b}\Delta\tau_i) > \frac{u}{3} \right).
\end{aligned} \tag{B.41}$$

It follows from $\mathsf{E}[\exp\{Q(\Delta\Omega_n)\}] < \infty$ ($n = 0, 1$) and $\eta x_*^2 \ge 1$ that for all x and u such that $\eta x_*^2 \le x \le \eta u^2$,

$$\mathsf{P}\left(\Delta\Omega_n > \frac{u}{3} \right) \le Ce^{-Q(u/3)} \le C\eta x_*^2 e^{-Q(u/3)} \le Cxe^{-Q(u/3)}, \qquad n = 0, 1,$$

from which and (B.41) we have

$$\mathsf{P}\left(\sup_{0\le t\le x}\{\breve{B}(t)-\breve{b}t\}>u\right)\le Cxe^{-Q(u/3)}+\mathsf{P}\left(\max_{1\le n\le N(x)}\sum_{i=1}^{n}(\Delta\breve{B}_i-\breve{b}\Delta\tau_i)>\frac{u}{3}\right). \quad \text{(B.42)}$$

We now fix $\delta>0$ arbitrarily and then have

$$\begin{aligned}&\mathsf{P}\left(\max_{1\le n\le N(x)}\sum_{i=1}^{n}(\Delta\breve{B}_i-\breve{b}\Delta\tau_i)>\frac{u}{3}\right)\\&\quad\le\mathsf{P}\left(N(x)-\frac{x}{\mathsf{E}[\Delta\tau_1]}>\delta x\right)+\mathsf{P}\left(\max_{1\le n\le(\delta+1/\mathsf{E}[\Delta\tau_1])x}\sum_{i=1}^{n}(\Delta\breve{B}_i-\breve{b}\Delta\tau_i)>\frac{u}{3}\right). \quad \text{(B.43)}\end{aligned}$$

Applying Lemma A.8 (which requires $\mathsf{E}[(\Delta\tau_1)^2]<\infty$) to the first term on the right hand side of (B.43), we have

$$\mathsf{P}\left(N(x)-\frac{x}{\mathsf{E}[\Delta\tau_1]}>\delta x\right)\le Ce^{-cx},\qquad x\ge 0. \quad \text{(B.44)}$$

Note that $\Delta\Omega_1\ge 0$ and $\Delta\Omega_1\ge\Delta\breve{B}_1-\breve{b}\Delta\tau_1$, which leads to $\Delta\Omega_1\ge(\Delta\breve{B}_1-\breve{b}\Delta\tau_1)^+$. Thus $\mathsf{E}[\exp\{Q(\Delta\Omega_1)\}]<\infty$ yields

$$\mathsf{E}[\exp\{Q((\Delta\breve{B}_1-\breve{b}\Delta\tau_1)^+)\}]<\infty.$$

Further, it follows from $\mathsf{E}[(\Delta\tau_1)^2]<\infty$, $\mathsf{E}[(\Delta\breve{B}_1)^2]<\infty$ and Hölder's inequality (see, e.g., [44]) that

$$\left|\mathsf{E}[\Delta\breve{B}_1\Delta\tau_1]\right|\le\sqrt{\mathsf{E}[(\Delta\breve{B}_1)^2]}\sqrt{\mathsf{E}[(\Delta\tau_1)^2]}<\infty,$$

which implies $\mathsf{E}[(\Delta\breve{B}_1-\breve{b}\Delta\tau_1)^2]<\infty$.

We now need the following result:

Proposition B.2 (Lemma 5 in [25]‡) *Let U_i's ($i=1,2,\dots$) denote independent copies of an r.v. U in $\mathbb{R}$. If $\mathsf{E}[U^2]<\infty$ and $\mathsf{E}[e^{Q(U^+)}]<\infty$ for some $Q\in\mathcal{SC}$, then for all $x,u\ge 0$,*

$$\mathsf{P}\left(\max_{1\le n\le x}\left\{\sum_{i=1}^{n}U_i-n\mathsf{E}[U]\right\}>u\right)\le C\left(e^{-cu^2/x}+xe^{-(1/2)Q(u)}\right),$$

where C and c are independent of x and u.

Applying Proposition B.2 to the second term on the right hand side of (B.43) and using $\mathsf{E}[\Delta\breve{B}_1-\breve{b}\Delta\tau_1]=0$, we obtain

$$\mathsf{P}\left(\max_{1\le n\le(\delta+1/\mathsf{E}[\Delta\tau_1])x}\sum_{i=1}^{n}(\Delta\breve{B}_i-\breve{b}\Delta\tau_i)>\frac{u}{3}\right)\le C\left(e^{-cu^2/x}+xe^{-(1/2)Q(u/3)}\right). \quad \text{(B.45)}$$

‡Although $\mathsf{E}[U^2]<\infty$ is not explicitly assumed in Lemma 5 in [25], this condition is required to prove the lemma (see p. 110 therein).

Substituting (B.44) and (B.45) into (B.43) yields

$$\mathsf{P}\left(\max_{1\le n\le N(x)}\sum_{i=1}^{n}(\Delta\breve{B}_i-\breve{b}\Delta\tau_i)>\frac{u}{3}\right)\le C\left(e^{-cx}+e^{-cu^2/x}+xe^{-(1/2)Q(u/3)}\right),$$

from which and (B.42), we have

$$\mathsf{P}\left(\sup_{0\le t\le x}\{\breve{B}(t)-\breve{b}t\}>u\right)\le C\left(e^{-cx}+e^{-cu^2/x}+xe^{-(1/2)Q(u/3)}\right). \tag{B.46}$$

Recall that in case (c), $u\ge x_*$ and therefore $Q(u/3)\ge(1/3)Q(u)$ due to (B.40). Finally, (B.46) yields (B.1).

B.17 Proof of Proposition B.1

We first prove (B.9). The Taylor expansion of $(x+y)^\gamma$ is given by

$$(x+y)^\gamma=\sum_{n=0}^{\infty}\frac{\gamma(\gamma-1)\cdots(\gamma-n+1)}{n!}y^n x^{\gamma-n},$$

from which we have

$$(x+y)^\gamma\le x^\gamma+\sum_{n=1}^{\infty}\frac{|\gamma(\gamma-1)\cdots(\gamma-n+1)|}{n!}\left(\frac{y}{x}\right)^{n-1}yx^{\gamma-1}. \tag{B.47}$$

Note here that

$$\begin{aligned}|\gamma(\gamma-1)\cdots(\gamma-n+1)|&=\gamma(\gamma-1)\cdots(\gamma-\lfloor\gamma\rfloor)\cdot\prod_{i=1}^{n-1-\lfloor\gamma\rfloor}(i+\lfloor\gamma\rfloor-\gamma)\\&\le\gamma(\gamma-1)\cdots(\gamma-\lfloor\gamma\rfloor)\cdot n!.\end{aligned}$$

Thus since $y/x<1$,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{|\gamma(\gamma-1)\cdots(\gamma-n+1)|}{n!}\left(\frac{y}{x}\right)^{n-1}&\le\gamma(\gamma-1)\cdots(\gamma-\lfloor\gamma\rfloor)\cdot\sum_{n=1}^{\infty}\left(\frac{y}{x}\right)^{n-1}\\&=\gamma(\gamma-1)\cdots(\gamma-\lfloor\gamma\rfloor)\cdot\left(1-\frac{y}{x}\right)^{-1}.\end{aligned} \tag{B.48}$$

Substituting (B.48) into (B.47) yields (B.9).

In the same way as the proof of (B.9), we have

$$(x-y)^\gamma\ge x^\gamma-\sum_{n=1}^{\infty}\frac{|\gamma(\gamma-1)\cdots(\gamma-n+1)|}{n!}\left(\frac{y}{x}\right)^{n-1}yx^{\gamma-1},$$

from which and (B.48) we obtain (B.10).

C Proofs of Main Results

C.1 Proof of Theorem 3.1

Using the monotonicity of $\{B(t)\}$, we have for $x \ge 1$,

$$\begin{aligned}
\mathsf{P}(B(T) > x) &\le \mathsf{P}(T > x - x^{2/3}) + \mathsf{P}(B(T) > x, T \le x - x^{2/3}) \\
&\le \mathsf{P}(T > x - x^{2/3}) + \mathsf{P}(B(x - x^{2/3}) > x), \\
\mathsf{P}(B(T) > x) &\ge \mathsf{P}(B(T) > x, T > x + x^{2/3}) \\
&= \mathsf{P}(T > x + x^{2/3}) - \mathsf{P}(B(T) \le x, T > x + x^{2/3}) \\
&\ge \mathsf{P}(T > x + x^{2/3}) - \mathsf{P}(B(x + x^{2/3}) \le x).
\end{aligned}$$

Further, since $T \in \mathcal{L}^3$, it follows from Lemma A.2 that

$$\mathsf{P}(T > x + x^{2/3}) \stackrel{x}{\sim} \mathsf{P}(T > x - x^{2/3}) \stackrel{x}{\sim} \mathsf{P}(T > x).$$

Thus it suffices to show that

$$\begin{aligned}
\mathsf{P}(B(x - x^{2/3}) > x) &= o(\mathsf{P}(T > x - x^{2/3})), \\
\mathsf{P}(B(x + x^{2/3}) \le x) &= o(\mathsf{P}(T > x + x^{2/3})).
\end{aligned}$$

For $x \ge 1$, we have

$$\begin{aligned}
\mathsf{P}(B(x - x^{2/3}) > x) &= \mathsf{P}\left(B(x - x^{2/3}) - (x - x^{2/3}) > x^{2/3}\right) \\
&\le \mathsf{P}\left(\sup_{0 \le t \le x - x^{2/3}} (B(t) - t) > x^{2/3}\right) \\
&\le \mathsf{P}\left(\sup_{0 \le t \le x} (B(t) - t) > x^{2/3}\right), \qquad \text{(C.1)} \\
\mathsf{P}(B(x + x^{2/3}) \le x) &\le \mathsf{P}(B(x + x^{2/3}) < x + (1/2)x^{2/3}) \\
&= \mathsf{P}\left(B(x + x^{2/3}) - (x + x^{2/3}) < -(1/2)x^{2/3}\right) \\
&\le \mathsf{P}\left(\inf_{0 \le t \le x + x^{2/3}} (B(t) - t) < -(1/2)x^{2/3}\right) \\
&\le \mathsf{P}\left(\inf_{0 \le t \le 2x} (B(t) - t) < -(1/2)x^{2/3}\right). \qquad \text{(C.2)}
\end{aligned}$$

Applying Lemma 2.1 (i) to the right hand side of (C.1), we obtain

$$\mathsf{P}(B(x - x^{2/3}) > x) \le C\left(e^{-cx^{1/3}} + e^{-cx} + xe^{-cQ(x^{2/3})}\right), \qquad x \ge 1. \qquad \text{(C.3)}$$

Since $T^\theta \in \mathcal{L}$, we have $\mathsf{P}(T > x) = e^{-o(x^\theta)}$ (see Lemma A.1) and thus for any $0 < \theta \le 1/3$,

$$\begin{aligned}
\limsup_{x\to\infty} \frac{e^{-cx^{1/3}}}{\mathsf{P}(T > x - x^{2/3})} &= \limsup_{x\to\infty} e^{-cx^{1/3} + o(x^\theta)} = 0, \\
\limsup_{x\to\infty} \frac{e^{-cx}}{\mathsf{P}(T > x - x^{2/3})} &= \limsup_{x\to\infty} e^{-cx + o(x^\theta)} = 0.
\end{aligned}$$

Further, it follows from (3.1) and $\mathsf{P}(T > x) = e^{-o(x^\theta)}$ that

$$\limsup_{x\to\infty} \frac{xe^{-cQ(x^{2/3})}}{\mathsf{P}(T > x - x^{2/3})} \le \limsup_{x\to\infty} e^{-cx^\theta + \log x + o(x^\theta)} = 0.$$

As a result, we have $\mathsf{P}(B(x - x^{2/3}) > x) = o(\mathsf{P}(T > x - x^{2/3}))$.

As for $\mathsf{P}(B(x + x^{2/3}) \le x)$, applying Lemma 2.1 (ii) to the right hand side of (C.2) yields

$$\mathsf{P}(B(x + x^{2/3}) \le x) \le C\left(e^{-cx^{1/3}} + e^{-cx} + xe^{-cQ((1/2)x^{2/3})}\right).$$

Therefore, similarly to the estimation of (C.3), we can readily show $\mathsf{P}(B(x + x^{2/3}) \le x) = o(\mathsf{P}(T > x + x^{2/3}))$. □

C.2 Proof of Theorem 3.2

We fix $\varepsilon \in (0,1)$ arbitrarily. For $x > 0$, we have

$$\begin{aligned}
\mathsf{P}(B(T) > x) &\le \mathsf{P}(T > (1-\varepsilon)x) + \mathsf{P}(B(T) > x, T \le (1-\varepsilon)x) \\
&\le \mathsf{P}(T > (1-\varepsilon)x) + \mathsf{P}(B((1-\varepsilon)x) > x), \\
\mathsf{P}(B(T) > x) &\ge \mathsf{P}(B(T) > x, T > (1+\varepsilon)x) \\
&= \mathsf{P}(T > (1+\varepsilon)x) - \mathsf{P}(B(T) \le x, T > (1+\varepsilon)x) \\
&\ge \mathsf{P}(T > (1+\varepsilon)x) - \mathsf{P}(B((1+\varepsilon)x) \le x).
\end{aligned}$$

Since $T \in \mathcal{C}$,

$$\lim_{\varepsilon\downarrow 0} \liminf_{x\to\infty} \frac{\mathsf{P}(T > (1+\varepsilon)x)}{\mathsf{P}(T > x)} = 1, \tag{C.4}$$

$$\lim_{\varepsilon\downarrow 0} \limsup_{x\to\infty} \frac{\mathsf{P}(T > (1-\varepsilon)x)}{\mathsf{P}(T > x)} = 1. \tag{C.5}$$

Therefore, it suffices to show that

$$\mathsf{P}(B((1-\varepsilon)x) > x) = o(\mathsf{P}(T > x)), \tag{C.6}$$

$$\mathsf{P}(B((1+\varepsilon)x) \le x) = o(\mathsf{P}(T > x)). \tag{C.7}$$

For $x > 0$, we have

$$\begin{aligned}
\mathsf{P}(B((1-\varepsilon)x) > x) &= \mathsf{P}(B((1-\varepsilon)x) - (1-\varepsilon)x > \varepsilon x) \\
&\le \mathsf{P}\left(\sup_{0\le t\le(1-\varepsilon)x} (B(t) - t) > \varepsilon x\right),
\end{aligned} \tag{C.8}$$

$$\begin{aligned}
\mathsf{P}(B((1+\varepsilon)x) \le x) &= \mathsf{P}(B((1+\varepsilon)x) - (1+\varepsilon)x \le -\varepsilon x) \\
&\le \mathsf{P}(B((1+\varepsilon)x) - (1+\varepsilon)x < -\varepsilon x/2) \\
&\le \mathsf{P}\left(\sup_{0\le t\le(1+\varepsilon)x} (t - B(t)) > \varepsilon x/2\right).
\end{aligned}$$

Let $N(t) = \max\{n \ge 0; \sum_{i=1}^{n} \Delta\tau_i \le t\}$ for $t \ge 0$. We then have

$$B(t) - t \le \Delta B_0 + \Delta B_{N(t-\tau_0)+1} + \sum_{i=1}^{N(t-\tau_0)} (\Delta B_i - \Delta\tau_i), \tag{C.9}$$
$$t - B(t) \le \Delta\tau_0 + \Delta\tau_{N(t-\tau_0)+1} + \sum_{i=1}^{N(t-\tau_0)} (\Delta\tau_i - \Delta B_i).$$

Note here that $N(t - \tau_0) \le N(t) \le N((1 - \varepsilon)x)$ for $0 \le t \le (1 - \varepsilon)x$. It thus follows from (C.8) and (C.9) that

$$\begin{aligned}\mathsf{P}(B((1-\varepsilon)x) > x) &\le \mathsf{P}(\Delta B_0 > \varepsilon x/3) + \mathsf{P}(\Delta B_1 > \varepsilon x/3) \\ &\quad + \mathsf{P}\left(\max_{1\le k\le N((1-\varepsilon)x)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon}{3}x\right), \quad x > 0, \end{aligned} \tag{C.10}$$

where we use the inequality $\mathsf{P}(X^{(1)} + X^{(2)} + X^{(3)} > x) \le \sum_{m=1}^{3} \mathsf{P}(X^{(m)} > x/3)$ for any triple of r.v.s $X^{(m)}$'s ($m = 1, 2, 3$). Similarly,

$$\begin{aligned}\mathsf{P}(B((1+\varepsilon)x) \le x) &\le \mathsf{P}(\Delta\tau_0 > \varepsilon x/6) + \mathsf{P}(\Delta\tau_1 > \varepsilon x/6) \\ &\quad + \mathsf{P}\left(\max_{1\le k\le N((1+\varepsilon)x)} \sum_{i=1}^{k} (\Delta\tau_i - \Delta B_i) > \frac{\varepsilon}{6}x\right), \quad x > 0. \end{aligned} \tag{C.11}$$

Further, since $T \in \mathcal{C} \subset \mathcal{D}$, condition (iii) yields

$$\lim_{x\to\infty} \frac{\mathsf{P}(\Delta B_n > \varepsilon x/3)}{\mathsf{P}(T > x)} \le \limsup_{x\to\infty} \frac{\mathsf{P}(\Delta B_n > \varepsilon x/3)}{\mathsf{P}(T > \varepsilon x/3)} \limsup_{x\to\infty} \frac{\mathsf{P}(T > \varepsilon x/3)}{\mathsf{P}(T > x)} = 0,$$
$$\lim_{x\to\infty} \frac{\mathsf{P}(\Delta\tau_n > \varepsilon x/6)}{\mathsf{P}(T > x)} \le \limsup_{x\to\infty} \frac{\mathsf{P}(\Delta\tau_n > \varepsilon x/6)}{\mathsf{P}(T > \varepsilon x/6)} \limsup_{x\to\infty} \frac{\mathsf{P}(T > \varepsilon x/6)}{\mathsf{P}(T > x)} = 0.$$

As for the third terms in (C.10) and (C.11), they are estimated as follows: for any $\delta > 0$,

$$\begin{aligned}&\mathsf{P}\left(\max_{1\le k\le N((1-\varepsilon)x)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon}{3}x\right) \\ &\quad \le \mathsf{P}\left(N((1-\varepsilon)x) - \frac{(1-\varepsilon)x}{\mathsf{E}[\Delta\tau_1]} > \delta x\right) \\ &\qquad + \mathsf{P}\left(N((1-\varepsilon)x) - \frac{(1-\varepsilon)x}{\mathsf{E}[\Delta\tau_1]} \le \delta x, \max_{1\le k\le N((1-\varepsilon)x)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon}{3}x\right) \\ &\quad \le \mathsf{P}\left(N((1-\varepsilon)x) - \frac{(1-\varepsilon)x}{\mathsf{E}[\Delta\tau_1]} > \delta x\right) \\ &\qquad + \mathsf{P}\left(\max_{1\le k\le \{\delta + (1-\varepsilon)/\mathsf{E}[\Delta\tau_1]\}x} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon}{3}x\right), \qquad x > 0. \end{aligned} \tag{C.12}$$

Similarly, for any $\delta > 0$,

$$\begin{aligned}
&\mathsf{P}\left(\max_{1 \le k \le N((1+\varepsilon)x)} \sum_{i=1}^{k} (\Delta\tau_i - \Delta B_i) > \frac{\varepsilon}{6} x \right) \\
&\quad \le \mathsf{P}\left(N((1+\varepsilon)x) - \frac{(1+\varepsilon)x}{\mathsf{E}[\Delta\tau_1]} > \delta x \right) \\
&\qquad + \mathsf{P}\left(\max_{1 \le k \le \{\delta + (1+\varepsilon)/\mathsf{E}[\Delta\tau_1]\}x} \sum_{i=1}^{k} (\Delta\tau_i - \Delta B_i) > \frac{\varepsilon}{6} x \right), \qquad x > 0. \qquad \text{(C.13)}
\end{aligned}$$

According to Lemma A.8, the first terms in (C.12) and (C.13) are bounded from above by $Ce^{-cx} = o(\mathsf{P}(T > x))$. As a result, to prove (C.6) and (C.7), it remains to show

$$\mathsf{P}\left(\max_{1 \le k \le \{\delta + (1-\varepsilon)/\mathsf{E}[\Delta\tau_1]\}x} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon}{3} x \right) = o(\mathsf{P}(T > x)), \qquad \text{(C.14)}$$

$$\mathsf{P}\left(\max_{1 \le k \le \{\delta + (1+\varepsilon)/\mathsf{E}[\Delta\tau_1]\}x} \sum_{i=1}^{k} (\Delta\tau_i - \Delta B_i) > \frac{\varepsilon}{6} x \right) = o(\mathsf{P}(T > x)). \qquad \text{(C.15)}$$

In what follows, the proof of (C.14) and (C.15) is performed under condition (v.a) and condition (v.b), separately.

C.2.1 Condition (v.a)

Suppose condition (v.a) holds and fix γ such that

$$\begin{aligned}
\gamma &= \min\left(\frac{\varepsilon}{3} \cdot \frac{1}{\delta + (1-\varepsilon)/\mathsf{E}[\Delta\tau_1]},\ \frac{\varepsilon}{6} \cdot \frac{1}{\delta + (1+\varepsilon)/\mathsf{E}[\Delta\tau_1]} \right) \\
&= \frac{\varepsilon}{6} \cdot \frac{1}{\delta + (1+\varepsilon)/\mathsf{E}[\Delta\tau_1]}. \qquad \text{(C.16)}
\end{aligned}$$

It then follows from Lemma A.9 that for any fixed $p > 0$,

$$\mathsf{P}\left(\max_{1 \le k \le \varepsilon/(3\gamma)x} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon}{3} x \right) \le Cx\mathsf{P}(\Delta B_1 - \Delta\tau_1 > vx) + Cx^{-p},$$

where v is some finite positive constant depending on p. Further, from $T \in \mathcal{C} \subset \mathcal{D}$ and condition (iv), we have

$$\begin{aligned}
&\lim_{x\to\infty} \frac{x\mathsf{P}(\Delta B_1 - \Delta\tau_1 > vx)}{\mathsf{P}(T > x)} \\
&\quad \le \limsup_{x\to\infty} \frac{x\mathsf{P}(\Delta B_1 - \Delta\tau_1 > vx)}{\mathsf{P}(T > vx)} \limsup_{x\to\infty} \frac{\mathsf{P}(T > vx)}{\mathsf{P}(T > x)} = 0.
\end{aligned}$$

We now fix $p > J^+$, where J^+ denotes the upper Matuszewska index of the d.f. of $\Delta B_1 - \Delta\tau_1$ (see subsection 2.2). Proposition 2.1 implies that $x^{-p} = o(\mathsf{P}(\Delta B_1 - \Delta\tau_1 > x))$, and thus condition (iii) yields $x^{-p} = o(\mathsf{P}(T > x))$. Therefore, (C.14) holds. Similarly, we can prove (C.15).

C.2.2 Condition (v.b)

Suppose condition (v.b) holds and define Y as an r.v. in $\mathbb{R}$ such that

$$\mathsf{P}(Y > x) = \min(1, c\mathsf{P}(T > x)/x), \qquad x > 0.$$

It then follows from $T \in \mathcal{C}$ and conditions (iv) and (v.b) that

$$Y^+ \in \mathcal{C}, \quad \mathsf{E}[Y] < \infty, \quad \mathsf{P}(|\Delta B_1 - \Delta\tau_1| > x) = o(\mathsf{P}(Y > x)) = o(\mathsf{P}(T > x)/x).$$

Therefore, Lemma A.7 implies that there exists an r.v. Z in $\mathbb{R}$ such that $Z \ge |\Delta B_1 - \Delta\tau_1|$ w.p.1 and

$$0 < \mathsf{E}[Z] < \gamma, \;\; Z^+ \in \mathcal{C}, \;\; \mathsf{P}(Z > x) = o(\mathsf{P}(T > x)/x),$$

where γ is given in (C.16). Since $Z \ge |\Delta B_1 - \Delta\tau_1|$ w.p.1, we have

$$\begin{aligned}
&\mathsf{P}\left(\max_{1 \le k \le \{\delta + (1-\varepsilon)/\mathsf{E}[\Delta\tau_1]\}x} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon}{3}x \right) \\
&\qquad \le \mathsf{P}\left(\max_{1 \le k \le \varepsilon x/(3\gamma)} \sum_{i=1}^{k} Z_i > \varepsilon x/3 \right),
\end{aligned}$$

where the Z_i's ($i = 1, 2, \dots$) are independent copies of Z. Applying Lemma A.10 to the right hand side of the above inequality yields

$$\mathsf{P}\left(\max_{1 \le k \le \{\delta + (1-\varepsilon)/\mathsf{E}[\Delta\tau_1]\}x} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon}{3}x \right) \le Cx\mathsf{P}(Z > \varepsilon x/3).$$

Further, it follows from $T \in \mathcal{C} \subset \mathcal{D}$ and $\mathsf{P}(Z > x) = o(\mathsf{P}(T > x)/x)$ that

$$\lim_{x\to\infty} \frac{x\mathsf{P}(Z > \varepsilon x/3)}{\mathsf{P}(T > x)} \le \limsup_{x\to\infty} \frac{x\mathsf{P}(Z > \varepsilon x/3)}{\mathsf{P}(T > \varepsilon x/3)} \limsup_{x\to\infty} \frac{\mathsf{P}(T > \varepsilon x/3)}{\mathsf{P}(T > x)} = 0.$$

Consequently, we have (C.14). As for the proof of (C.15), we can proceed as in the proof of (C.14).

C.3 Proof of Theorem 3.3

The asymptotic lower bound $\mathsf{P}(B(T) > x) \gtrsim_x \mathsf{P}(T > x)$ can be proved in the same way as the proof of Theorem 3 in [25]. Thus we here prove only the asymptotic upper bound $\mathsf{P}(M(T) > x) \lesssim_x \mathsf{P}(T > x)$.

We fix δ $(0 < \delta < 1)$ arbitrarily and also fix x such that $0 < \delta x \le x - \xi\sqrt{x}$ and $\xi \ge 1$, which leads to $\sqrt{x} \ge \xi/(1-\delta) > 1$. We then have

$$\begin{aligned}
\mathsf{P}(M(T) > x) &= \mathsf{P}(M(T) > x, T > x - \xi\sqrt{x}) \\
&\qquad + \mathsf{P}(M(T) > x, \delta x < T \le x - \xi\sqrt{x}) + \mathsf{P}(M(T) > x, T \le \delta x) \\
&\le \mathsf{P}(T > x - \xi\sqrt{x}) \\
&\qquad + \mathsf{P}(M(T) > x, \delta x < T \le x - \xi\sqrt{x}) + \mathsf{P}(M(\delta x) > x). \qquad \text{(C.17)}
\end{aligned}$$

Since $T \in \mathcal{L}^2$, $\mathsf{P}(T > x - \xi\sqrt{x}) \overset{x}{\sim} \mathsf{P}(T > x)$ (see Lemma A.2). Therefore, it suffices to show that the second and third terms in (C.17) are $o(\mathsf{P}(T > x))$.

Note first that

$$\begin{aligned}
\mathsf{P}(M(\delta x) > x) &= \mathsf{P}\left(\sup_{0\le t\le \delta x} B(t) - \delta x > (1-\delta)x\right) \\
&\le \mathsf{P}\left(\sup_{0\le t\le \delta x} (B(t) - t) > (1-\delta)x\right). \qquad \text{(C.18)}
\end{aligned}$$

Applying Lemma 2.1 (i) to (C.18) yields

$$\begin{aligned}
\mathsf{P}(M(\delta x) > x) &\le C\left(e^{-cx} + xe^{-cQ((1-\delta)x)}\right) \\
&= o(\mathsf{P}(T > x)) + Cxe^{-cQ((1-\delta)x)}.
\end{aligned}$$

Further, since $x^\theta = O(Q(x))$ and $\mathsf{P}(T > x) = e^{-o(x^\theta)}$,

$$\limsup_{x\to\infty} \frac{xe^{-cQ((1-\delta)x)}}{\mathsf{P}(T > x)} \le \limsup_{x\to\infty} \exp\left\{-cx^\theta/C + \log x + o(x^\theta)\right\} = 0.$$

Consequently, we have $\mathsf{P}(M(\delta x) > x) = o(\mathsf{P}(T > x))$.

Next, we consider the second term on the right hand side of (C.17). Note that

$$\begin{aligned}
&\mathsf{P}(M(T) > x, \delta x < T \le x - \xi\sqrt{x}) \\
&\quad = \int_{\delta x}^{x-\xi\sqrt{x}} \mathsf{P}(M(u) > x)\mathrm{d}\mathsf{P}(T \le u) \\
&\quad \le \int_{\delta x}^{x-\xi\sqrt{x}} \mathsf{P}\left(\sup_{0\le t\le u} (B(t) - t) > x - u\right) \mathrm{d}\mathsf{P}(T \le u). \qquad \text{(C.19)}
\end{aligned}$$

Applying Lemma 2.1 (i) to (C.19) and using $\delta x \le u \le x$, we obtain

$$\begin{aligned}
&\mathsf{P}(M(T) > x, \delta x < T \le x - \xi\sqrt{x}) \\
&\quad \le \int_{\delta x}^{x-\xi\sqrt{x}} C\left(e^{-c(x-u)^2/u} + e^{-cu} + ue^{-cQ(x-u)}\right) \mathrm{d}\mathsf{P}(T \le u) \\
&\quad \le Ce^{-c\delta x} + C\int_{\delta x}^{x-\xi\sqrt{x}} \left(e^{-c(x-u)^2/x} + xe^{-cQ(x-u)}\right) \mathrm{d}\mathsf{P}(T \le u) \\
&\quad = o(\mathsf{P}(T > x)) + Cf_1(x) + Cf_2(x),
\end{aligned}$$

where

$$f_1(x) = \int_{\delta x}^{x-\xi\sqrt{x}} e^{-c(x-u)^2/x}\mathrm{d}\mathsf{P}(T \le u), \qquad \text{(C.20)}$$

$$f_2(x) = \int_{\delta x}^{x-\xi\sqrt{x}} xe^{-cQ(x-u)}\mathrm{d}\mathsf{P}(T \le u). \qquad \text{(C.21)}$$

In what follows, we prove $f_1(x) = o(\mathsf{P}(T > x))$ and $f_2(x) = o(\mathsf{P}(T > x))$. Note that $e^{-c(x-u)^2/x}$ is continuous with respect to u. Thus integrating the right hand side of (C.20) by parts (see, e.g., Theorems 6.1.7 and 6.2.2 in [9]) and letting $y = (x-u)/\sqrt{x}$ yield

$$\begin{aligned}
f_1(x) &\le e^{-c(1-\delta)^2 x} + \int_{\delta x}^{x-\xi\sqrt{x}} \mathsf{P}(T > u) \mathrm{d}(e^{-c(x-u)^2/x}) \\
&= e^{-c(1-\delta)^2 x} + \int_{\xi}^{(1-\delta)\sqrt{x}} \mathsf{P}(T > x - y\sqrt{x}) 2cye^{-cy^2} \mathrm{d}y \\
&= o(\mathsf{P}(T > x)) + \int_{\xi}^{(1-\delta)\sqrt{x}} \mathsf{P}(T > x - y\sqrt{x}) 2cye^{-cy^2} \mathrm{d}y \\
&\le o(\mathsf{P}(T > x)) + \int_{\xi}^{(1-\delta)\sqrt{x}} \mathsf{P}(\sqrt{T} > \sqrt{x} - y) 2cye^{-cy^2} \mathrm{d}y,
\end{aligned} \tag{C.22}$$

where the last inequality holds because $(x - y\sqrt{x})^{1/2} \ge \sqrt{x} - y$ for $0 \le y \le \sqrt{x}$. It thus follows from Lemma A.4 that for any $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{\xi\to\infty} \limsup_{x\to\infty} \int_{\xi}^{(1-\delta)\sqrt{x}} \frac{\mathsf{P}(\sqrt{T} > \sqrt{x} - y)}{\mathsf{P}(T > x)} 2cye^{-cy^2} \mathrm{d}y \\
&\quad = \lim_{\xi\to\infty} \limsup_{x\to\infty} \int_{\xi}^{(1-\delta)\sqrt{x}} \frac{\mathsf{P}(\sqrt{T} > \sqrt{x} - y)}{\mathsf{P}(\sqrt{T} > \sqrt{x})} 2cye^{-cy^2} \mathrm{d}y \\
&\quad \le e^{\varepsilon} \lim_{\xi\to\infty} \limsup_{x\to\infty} \int_{\xi}^{(1-\delta)\sqrt{x}} 2cy \exp\{-cy^2 + \varepsilon y\} \mathrm{d}y \\
&\quad \le e^{\varepsilon} \lim_{\xi\to\infty} \int_{\xi}^{\infty} 2cy \exp\{-cy^2 + \varepsilon y\} \mathrm{d}y = 0.
\end{aligned} \tag{C.23}$$

Finally, combining (C.22) with (C.23) yields $f_1(x) = o(\mathsf{P}(T > x))$.

We proceed to the proof of $f_2(x) = o(\mathsf{P}(T > x))$. Note first that since Q is eventually concave (see Definition 2.2), Q is continuous for all sufficiently large $x > 0$ and differentiable for all sufficiently large $x > 0$ except countable points. In what follows, we fix x to be sufficiently large such that $Q(x-u)$ has these properties for all $\delta x \le u \le x - \xi\sqrt{x}$.

For $\delta x \le u \le x - \xi\sqrt{x}$, we have

$$e^{-cQ(x-u)} = e^{-(c/2)Q(x-u)} e^{-(c/2)Q(x-u)} \le e^{-(c/2)Q(\xi\sqrt{x})} e^{-(c/2)Q(x-u)}.$$

It thus follows from (C.21) that

$$f_2(x) \le xe^{-cQ(\xi\sqrt{x})} \int_{\delta x}^{x-\xi\sqrt{x}} \{-e^{-cQ(x-u)}\} \mathrm{d}\mathsf{P}(T > u).$$

Letting $y = x - u$ and integrating the right hand side by parts and change of valuables (see,

e.g., Theorems 6.2.1 and 6.2.2 in [9]), we have

$$\begin{aligned}
f_2(x) &\le xe^{-cQ(\xi\sqrt{x})}\left[e^{-cQ((1-\delta)x)} + \int_{\delta x}^{x-\xi\sqrt{x}} \mathsf{P}(T > u)\mathrm{d}(e^{-cQ(x-u)})\right] \\
&= xe^{-cQ(\xi\sqrt{x})}\left[e^{-cQ((1-\delta)x)} + \int_{\xi\sqrt{x}}^{(1-\delta)x} \mathsf{P}(T > x-y)\mathrm{d}(-e^{-cQ(y)})\right]. \qquad \text{(C.24)}
\end{aligned}$$

In estimating the right hand side of (C.24), we assume that Q is differentiable, without loss of generality. Under this assumption, (A.2) holds for some $0 < \alpha < 1$ (see Remark A.3). Thus we obtain

$$f_2(x) \le xe^{-cQ(\xi\sqrt{x})}\left[e^{-cQ((1-\delta)x)} + C\int_{\xi\sqrt{x}}^{(1-\delta)x} \mathsf{P}(T > x-y)e^{-cQ(y)}\mathrm{d}y\right]. \qquad \text{(C.25)}$$

It follows from $T^\theta \in \mathcal{L}$ and Lemma A.4 that for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
&\limsup_{x\to\infty}\int_{\xi\sqrt{x}}^{(1-\delta)x} \frac{\mathsf{P}(T > x-y)}{\mathsf{P}(T > x)}e^{-cQ(y)}\mathrm{d}y \\
&\qquad \le e^{\varepsilon}\limsup_{x\to\infty}\int_{\xi\sqrt{x}}^{(1-\delta)x} e^{\varepsilon y^\theta - cQ(y)}\mathrm{d}y = 0,
\end{aligned}$$

where the last equality is due to $x^\theta = O(Q(x))$. Further, using $\log x = o(Q(x))$ and $x^\theta = O(Q(x))$, we have

$$\begin{aligned}
&\lim_{x\to\infty} xe^{-cQ(\xi\sqrt{x})} = \lim_{x\to\infty} e^{-cQ(\xi\sqrt{x})+2\log\sqrt{x}} = \lim_{x\to\infty} e^{-cQ(\xi\sqrt{x})+o(Q(\xi\sqrt{x}))} = 0, \\
&\lim_{x\to\infty}\frac{e^{-cQ((1-\delta)x}}{\mathsf{P}(T > x)} \le \limsup_{x\to\infty} e^{-cx^\theta/C + o(x^\theta)} = 0.
\end{aligned}$$

As a result, the right hand side of (C.25) is $o(\mathsf{P}(T > x))$. □

C.4 Proof of Theorem 3.4

For any $\varepsilon > 0$, we have

$$\begin{aligned}
\mathsf{P}(B(T) > x) &\ge \int_{(1+\varepsilon)x}^{\infty} \mathsf{P}(B(u) > x)\mathrm{d}\mathsf{P}(T \le u) \\
&\ge \inf_{u>(1+\varepsilon)x} \mathsf{P}(B(u) > x)\mathsf{P}(T > (1+\varepsilon)x) \\
&= \inf_{u>(1+\varepsilon)x} \mathsf{P}\left(\frac{B(u)-u}{u} > \frac{x-u}{u}\right)\mathsf{P}(T > (1+\varepsilon)x) \\
&\ge \inf_{u>(1+\varepsilon)x} \mathsf{P}\left(\frac{B(u)-u}{u} > \frac{-\varepsilon}{1+\varepsilon}\right)\mathsf{P}(T > (1+\varepsilon)x). \qquad \text{(C.26)}
\end{aligned}$$

It follows from the SLLN for $\{B(t)\}$ (see [3, Chapter VI, Theorem 3.1]) that for any $\varepsilon > 0$,

$$\lim_{x\to\infty} \inf_{u>(1+\varepsilon)x} \mathsf{P}\left(\frac{B(u)-u}{u} > \frac{-\varepsilon}{1+\varepsilon} \right) \ge \lim_{x\to\infty} \inf_{u>(1+\varepsilon)x} \mathsf{P}\left(\left| \frac{B(u)-u}{u} \right| < \frac{\varepsilon}{1+\varepsilon} \right) = 1.$$

Note here that (C.4) holds due to $T \in \mathcal{C}$. Thus, from (C.26), we have $\mathsf{P}(B(T) > x) \gtrsim_x \mathsf{P}(T > x)$.

In what follows, we prove $\mathsf{P}(M(T) > x) \lesssim_x \mathsf{P}(T > x)$. For any $\varepsilon \in (0,1)$,

$$\mathsf{P}(M(T) > x) \le \mathsf{P}(T > (1-\varepsilon)x) + \mathsf{P}(M(T) > x, T \le (1-\varepsilon)x).$$

Since (C.5) holds, it suffices to show $\mathsf{P}(M(T) > x, T \le (1-\varepsilon)x) = o(\mathsf{P}(T > x))$.

For $x > 0$, we have

$$\begin{aligned}
&\mathsf{P}(M(T) > x, T \le (1-\varepsilon)x) \\
&\quad \le \int_0^{(1-\varepsilon)x} \mathsf{P}\left(\sup_{0\le t\le u} \{B(t) - t\} > x - u \right) \mathrm{d}\mathsf{P}(T \le u) \\
&\quad \le \int_0^{(1-\varepsilon)x} \mathsf{P}\left(\sup_{0\le t\le u} \{B(t) - t\} > \varepsilon x \right) \mathrm{d}\mathsf{P}(T \le u).
\end{aligned}$$

Note here that for $0 \le u \le (1-\varepsilon)x$,

$$\begin{aligned}
&\mathsf{P}\left(\sup_{0\le t\le u} \{B(t) - t\} > \varepsilon x \right) \\
&\quad \le \mathsf{P}\left(\Delta B_0^* + \Delta B_{N(u-\tau_0)+1}^* + \max_{1\le k\le N(u-\tau_0)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \varepsilon x \right) \\
&\quad \le \mathsf{P}\left(\Delta B_0^* > \varepsilon x/3\right) + \mathsf{P}\left(\Delta B_1^* > \varepsilon x/3\right) \\
&\qquad + \mathsf{P}\left(\max_{1\le k\le N(u)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon x}{3} \right),
\end{aligned}$$

where $N(t) = \max\{n \ge 0; \sum_{i=1}^{n} \Delta\tau_i \le t\}$ for $t \ge 0$. According to conditions (i) and (iii),

$$\mathsf{P}\left(\Delta B_n^* > \varepsilon x/3\right) = o(\mathsf{P}(T > x)), \quad n = 0, 1.$$

Therefore, the proof will be complete once we show that

$$\int_0^{(1-\varepsilon)x} \mathsf{P}\left(\max_{1\le k\le N(u)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon x}{3} \right) \mathrm{d}\mathsf{P}(T \le u) = o(\mathsf{P}(T > x)). \tag{C.27}$$

Fix a positive number γ such that

$$\frac{1-\varepsilon}{\mathsf{E}[\Delta\tau_1]} < \frac{\varepsilon}{3\gamma} \le \frac{1}{\mathsf{E}[\Delta\tau_1]}. \tag{C.28}$$

We then decompose the right hand side of (C.27) into $R_1(x) + R_2(x)$ in the following way:

$$\begin{aligned} R_1(x) &= \int_0^{(1-\varepsilon)x} \mathrm{d}\mathsf{P}(T \le u) \\ &\quad \times \mathsf{P}\left(\max_{1\le k\le N(u)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon x}{3},\ N(u) > \frac{\varepsilon x}{3\gamma} \right), \\ R_2(x) &= \int_0^{(1-\varepsilon)x} \mathrm{d}\mathsf{P}(T \le u) \\ &\quad \times \mathsf{P}\left(\max_{1\le k\le N(u)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon x}{3},\ N(u) \le \frac{\varepsilon x}{3\gamma} \right). \end{aligned} \tag{C.29}$$

For $x > 0$, we have

$$R_1(x) \le \int_0^{(1-\varepsilon)x} \mathsf{P}\left(N(u) > \frac{\varepsilon x}{3\gamma} \right) \mathrm{d}\mathsf{P}(T \le u) \le \mathsf{P}\left(N((1-\varepsilon)x) > \frac{\varepsilon x}{3\gamma} \right). \tag{C.30}$$

Note here that $\varepsilon/(3\gamma) - (1-\varepsilon)/\mathsf{E}[\Delta\tau_1] > 0$ due to (C.28). Thus Lemma A.8 yields

$$\begin{aligned} &\mathsf{P}\left(N((1-\varepsilon)x) > \frac{\varepsilon x}{3\gamma} \right) \\ &\quad = \mathsf{P}\left(N((1-\varepsilon)x) - \frac{(1-\varepsilon)x}{\mathsf{E}[\Delta\tau_1]} > \left(\frac{\varepsilon}{3\gamma} - \frac{1-\varepsilon}{\mathsf{E}[\Delta\tau_1]} \right) x \right) \\ &\quad \le Ce^{-cx} = o(\mathsf{P}(T > x)). \end{aligned}$$

Combining this with (C.30), we have $R_1(x) = o(\mathsf{P}(T > x))$.

As for $R_2(x)$, it follows from (C.29) that

$$R_2(x) \le \int_0^{(1-\varepsilon)x} \mathrm{d}\mathsf{P}(T \le u) \mathsf{P}\left(\max_{1\le k\le \varepsilon x/(3\gamma)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon x}{3} \right).$$

Similarly to the proof of (C.14), we can prove

$$\mathsf{P}\left(\max_{1\le k\le \varepsilon x/(3\gamma)} \sum_{i=1}^{k} (\Delta B_i - \Delta\tau_i) > \frac{\varepsilon x}{3} \right) = o(\mathsf{P}(T > x)),$$

which leads to $R_2(x) = o(\mathsf{P}(T > x))$. □

C.5 Proof of Theorem 3.5

We first confirm that the assumptions of Theorem 3.5 imply conditions (i), (ii) and (iii) of Theorem 3.4. For simplicity, we assume $h = b = 1$, which does not lose generality.

We now introduce a cumulative process $\{B^+(t); t \ge 0\}$ such that

$$B^+(t) = \sum_{n=0}^{\lfloor t \rfloor} |X_n|, \qquad t \ge 0.$$

Clearly, the regenerative points τ_n's of $\{B(t)\}$ are those of $\{B^+(t)\}$, and $\{(B^+(n), J_n); n = 0, 1, \dots\}$ is a Markov additive process with initial distribution $\boldsymbol{\beta}^+(x)$ and Markov additive kernel $\boldsymbol{H}^+(x)$ ($x \in \mathbb{R}$), where $\boldsymbol{\beta}^+(x) = \int_{|y| \le x} \mathrm{d}\boldsymbol{\beta}(y)$ and $\boldsymbol{H}^+(x) = \int_{|y| \le x} \mathrm{d}\boldsymbol{H}(y)$.

Let $\Delta B_n^+ = B^+(\tau_n) - B^+(\tau_{n-1})$ for $n = 0, 1, \dots$. We then have

$$\Delta B_n^+ \ge \sup_{\tau_{n-1} \le t \le \tau_n} |B(t) - B(\tau_{n-1})| \ge \Delta B_n^* \ge \Delta B_n.$$

For ΔB_n^+, we can readily prove the results corresponding to Proposition 3.1 and Lemma 3.1 with $Y = T$ and $c^* = 0$. Thus,

$$\mathsf{E}\left[\sup_{\tau_0 \le t \le \tau_1} |B(t) - B(\tau_0)|\right] \le \mathsf{E}[\Delta B_1^+] = \boldsymbol{\varpi} \int_{x \in \mathbb{R}} |x| \mathrm{d}\boldsymbol{H}(x) \boldsymbol{e} \cdot \mathsf{E}[\Delta \tau_1],$$

which is finite due to Assumption 3.1 (iii). Therefore, the SLLN holds for $\{B(t)\}$. Further, we obtain

$$\mathsf{P}(\Delta B_n^* > x) \le \mathsf{P}(\Delta B_n^+ > x) = o(\mathsf{P}(T > x)).$$

Recall here that the distribution of $\Delta \tau_n$ is of phase-type and thus $\mathsf{E}[(\Delta \tau_n)^2] < \infty$ ($n = 0, 1$).

The above argument shows that condition (i), (ii) and (iii) of Theorem 3.4 are satisfied. Thus we can prove $\mathsf{P}(B(T) > x) \gtrsim_x \mathsf{P}(T > x)$ in the same way as that of Theorem 3.4. As for the proof of $\mathsf{P}(M(T) > x) \lesssim_x \mathsf{P}(T > x)$, we also proceed as in that of Theorem 3.4, except for the estimation of $R_2(x)$ in (C.29).

As a result, it remains to prove that $R_2(x) = o(\mathsf{P}(T > x))$, which follows from (C.29) and Lemma 3.3 with $Y = T$ and $c^* = 0$. Indeed,

$$\begin{aligned}
R_2(x) &= \int_0^{(1-\varepsilon)x} \sum_{n \le \varepsilon x/(3\gamma)} \mathsf{P}(N(u) = n) \\
&\quad \times \mathsf{P}\left(\max_{1 \le k \le n} \sum_{i=1}^{k} (\Delta B_i - \Delta \tau_i) > \frac{\varepsilon x}{3} \,\middle|\, N(u) = n \right) \mathrm{d}\mathsf{P}(T \le u) \\
&\le \int_0^{(1-\varepsilon)x} \sum_{n \le \varepsilon x/(3\gamma)} \mathsf{P}(N(u) = n) \\
&\quad \times \mathsf{P}\left(\sum_{i=1}^{n} \Delta B_i > \frac{\varepsilon x}{3} \,\middle|\, N(u) = n \right) \mathrm{d}\mathsf{P}(T \le u) \\
&\le \int_0^{(1-\varepsilon)x} u \mathrm{d}\mathsf{P}(T \le u) \sum_{n \le \varepsilon x/(3\gamma)} \mathsf{P}(N(u) = n) \cdot o(\mathsf{P}(T > x)) \\
&\le \mathsf{E}[T] \cdot o(\mathsf{P}(T > x)).
\end{aligned}$$

C.6 Proof of Theorem 3.6

Since we obtain (3.6) from (3.7) and (3.8), the proof of Theorem 3.6 follows that of Theorem 3.5, except for the estimation of $R_2(x)$ in (C.29). As for the estimation of $R_2(x)$, we apply Lemma 3.3 to (C.29) and use (C.28). Consequently, we have

$$\begin{aligned} R_2(x) &\le C \int_0^{(1-\varepsilon)x} u \mathrm{d}\mathsf{P}(T \le u) \sum_{n \le \varepsilon x/(3\gamma)} \mathsf{P}(N(u) = n) \cdot \mathsf{P}(Y > x) \\ &\le C\mathsf{E}[T \cdot 1\!\!1(T \le x, N(T) \le x/\mathsf{E}[\Delta\tau_1])] \cdot \mathsf{P}(Y > x) \\ &= o(\mathsf{P}(T > x)), \end{aligned}$$

where the last equality is due to (3.8).

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